2015 - 2016 期终考试试卷答案.

一. 1. $A(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3)$

$= (\alpha_1, \alpha_2, \alpha_3)A$

基 $(\alpha_1, \cdots, \alpha_n)$ 到基 $(\beta_1, \cdots, \beta_n)$ 的过渡矩阵为T.
则 $(\alpha_1, \cdots, \alpha_n)T = (\beta_1, \cdots, \beta_n)$

$\therefore A = (\alpha_1, \alpha_2, \alpha_3)^{-1}(\beta_1, \beta_2, \beta_3)$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & -1 \\ 6 & 2 & 2 \end{pmatrix}$$

求逆:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-3r_1 \to r_3]{-2r_1 \to r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-r_2 \to r_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \quad 0 \quad 0 \\ 0 & 1 & 0 & -2 \quad 1 \quad 0 \\ 0 & 0 & 1 & \underbrace{-1 \quad -1 \quad 1}_{(\alpha_1, \alpha_2, \alpha_3)^{-1}} \end{array}\right)$$

$$\therefore A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & -1 \\ 6 & 2 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 3 & -1 \\ 6 & 2 & 3 \end{pmatrix}$$

2. $\cos\theta = \dfrac{\alpha \cdot \beta}{|\alpha| \cdot |\beta|}$

则 $\alpha \cdot \beta = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \cdot (\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4)$

$= \alpha_1 \cdot \alpha_2 + \alpha_2 \cdot \alpha_2 + \alpha_3 \cdot \alpha_3 - \alpha_4 \cdot \alpha_4 = 2.$

$|\alpha| = \sqrt{\alpha \cdot \alpha} = (\alpha_1 \cdot \alpha_1 + \alpha_2 \cdot \alpha_2 + \alpha_3 \cdot \alpha_3 + \alpha_4 \cdot \alpha_4)^{1/2} = 2$

$|\beta| = \sqrt{\beta \cdot \beta} = (\alpha_1 \cdot \alpha_1 + \alpha_2 \cdot \alpha_2 + \alpha_3 \cdot \alpha_3 + \alpha_4 \cdot \alpha_4)^{1/2} = 2.$

$\therefore \cos\theta = \dfrac{1}{2}$

$\therefore \theta = \dfrac{\pi}{3}.$

3. λ^m为A^m的特征值

$\because \lambda^m=0 \Rightarrow \lambda=0$

$\therefore$ A的特征值为0（n重）

$|\lambda I-A|=0$ 的解为$\lambda_1=\lambda_2=\cdots=\lambda_n=0$.

$\therefore |\lambda' I-(I+A)|=|(\lambda'-1)I-A|=0$ 的解为 $\lambda_1'=\lambda_2'=\cdots=\lambda_n'=1$

$\therefore \det(I+A)=1$

4. $\mathrm{diag}(1,2,3)$.

5. Q对应的矩阵 $A=\begin{pmatrix} 1 & -1 & 1 \\ -1 & a & 1 \\ 1 & 1 & 2 \end{pmatrix}$

正定 $\Leftrightarrow$ A的各阶顺序主子式均大于0.

$a_{11}=1>0$

$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a-1>0$

$\det(A)=a-5$

$\therefore \begin{cases} a-1>0 \\ a-5>0 \end{cases}$

故 正定 $\Leftrightarrow a>5$

二.

1.（√）$\forall \alpha,\beta\in V$，$\forall \mu\in F$

有 $\mathcal{A}(\alpha+\beta)=\mathcal{A}\alpha+\mathcal{A}\beta=\lambda(\alpha+\beta)$

$\mathcal{A}(\mu\alpha)=\mu\mathcal{A}(\alpha)=\lambda(\mu\alpha)$

$\therefore \alpha+\beta,\ \mu\alpha\in V_{\mathcal{A}}(\lambda)$

$\therefore V_{\mathcal{A}}(\lambda)$ 为 V 的子空间.

2.（√）设 $\lambda_1\alpha_1+\cdots+\lambda_m\alpha_m=0$

则 $0=(0,\alpha_i)=(\sum_{i=1}^{m}\lambda_i\alpha_i,\alpha_i)=\lambda_i(\alpha_i,\alpha_i)$

又 $\because \alpha_i\neq 0$，且 $(\alpha_i,\alpha_i)>0$，$\therefore \lambda_i=0$，$i=1,2,\cdots,m$.

3.（×）举反例，$A=\begin{pmatrix}-2&0\\0&-2\end{pmatrix}$，$t=1$，则 $A+I=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$，故不正定.

4.（√）显然 $Ax=0$ 的解是 $A^TAx=0$ 的解.

下证 $A^TAx=0$ 的解也是 $Ax=0$ 的解.

设 x 为 $A^TAx=0$ 的任一解，

则 $0=x^TA^TAx=(Ax)^TAx$

$\therefore Ax=0$，故得证.

三、(1) (15)

$$|\lambda I-A| = \begin{vmatrix} \lambda-1 & -2 & 0 \\ -1 & \lambda+2 & -1 \\ 0 & -2 & \lambda-1 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda+3)=0$$

$\therefore A$的特征值为 $\lambda_1=1, \lambda_2=2, \lambda_3=-3$.

5分

解方程组 $(I-A)x=0$，即 $\begin{pmatrix} 0 & -2 & 0 \\ -1 & 3 & -1 \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$，$\alpha_1=(1,0,-1)^T$

$(2I-A)x=0$，即 $\begin{pmatrix} 1 & -2 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$，$\alpha_2=(2,1,2)^T$

$(-3I-A)x=0$，即 $\begin{pmatrix} -4 & -2 & 0 \\ -1 & -1 & -1 \\ 0 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$，$\alpha_3=(1,-2,1)^T$

10分

令 $T=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{pmatrix}$

则 $T^{-1}AT=\mathrm{diag}(1,2,-3)$

12分

(2) $A^n=(T\,\mathrm{diag}(1,2,-3)T^{-1})^n$

$=T\,\mathrm{diag}(1,2^n,(-3)^n)\,T^{-1}$

$$=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & (-3)^n \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{10} & -\frac{2}{5} & \frac{1}{10} \end{pmatrix}$$

$$=\frac{1}{10}\begin{pmatrix} 5+4\cdot2^n+(-3)^n & 4\cdot2^n-4\cdot(-3)^n & -5+4\cdot2^n+(-3)^n \\ 2\cdot2^n-2\cdot(-3)^n & 2\cdot2^n+8\cdot(-3)^n & 2\cdot2^n-2\cdot(-3)^n \\ -5+4\cdot2^n+(-3)^n & 4\cdot2^n-4\cdot(-3)^n & 5+4\cdot2^n+(-3)^n \end{pmatrix}$$

15分

四.

(17) (1) $e_1=\frac{\alpha_1}{|\alpha_1|}=\frac{1}{\sqrt{3}}(1,1,1)$

$\beta_2=\alpha_2-(\alpha_2,e_1)e_1=\frac{1}{3}(-2,1,1)$, $e_2=\frac{\beta_2}{|\beta_2|}=\frac{1}{\sqrt{6}}(-2,1,1)$

$\beta_3=\alpha_3-(\alpha_3,e_2)e_2-(\alpha_3,e_1)e_1=\frac{1}{2}(0,-1,1)$, $e_3=\frac{\beta_3}{|\beta_3|}=\frac{1}{\sqrt{2}}(0,-1,1)$

12分

(2) $A=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$ 为正交矩阵，且 $\det(A)=1$，

由书 P205 命题 7.3.2，A 必有特征值 1.

故 $\mathcal{A}x=Ax$ 表示统特征值 1 的特征向量为轴的旋转变换.

15分

求解 $(I-A)x=0$，得特征向量 $\alpha=\left(\frac{(2-\sqrt{2})(\sqrt{3}+1)}{2},1-\sqrt{2},1\right)$

故 α 为旋转轴.

17分

I. 配方法

做变换 $\begin{cases}x'=x\\ y'=y+2z\\ z'=z\end{cases}$

得 $x'y'+2y'-2z'-1=0$

做变换 $\begin{cases}x'=x''+y''\\ y'=x''-y''\\ z'=z''\end{cases}$

得 $(x'')^2-(y'')^2+2x''-2y''+2z''-1=0$

配方得 $(x''+1)^2-(y''+1)^2+2z''-1=0$

做变换 $\begin{cases}\tilde{x}=x''+1\\ \tilde{y}=y''+1\\ \tilde{z}=-2z''+1\end{cases}$

得 $\tilde{x}^2-\tilde{y}^2=\tilde{z}$

为双曲抛物面.

答案不唯一，但惯性指数要正确

写出双曲面、抛物型中的一种扣1分

原方程化为 $\frac{\sqrt{5}}{2}(y')^2-\frac{\sqrt{5}}{2}(z')^2+\frac{2}{\sqrt{10}}(\sqrt{2}x'+3y'+3z')=1$

做平移变换 $\begin{cases}\tilde{x}=x'-\frac{1}{\sqrt{5}}\\ \tilde{y}=y'+\frac{3\sqrt{2}}{5}\\ \tilde{z}=z'-\frac{3\sqrt{2}}{5}\end{cases}$

则有 $-\frac{5}{2}\tilde{y}^2+\frac{5}{2}\tilde{z}^2=\tilde{x}$

故为双曲抛物面.

$A^2=I$

$\therefore$ A的特征值满足$\lambda^2=1$ （$Ax=\lambda x$, $A^2x=\lambda^2x=x$，则$\lambda^2=1$）

即 $\lambda=\pm1$

又$\because$ A为实对称方阵，故A正交相似于对角阵 $\mathrm{diag}(I^r,-I^{n-r})$

故$\exists$正交阵P, s.t. $A=P\cdot\mathrm{diag}(I^r,-I^{n-r})P^{-1}$

5分

$I+A=P(2I^r,0)P^{-1}$,

令 $B=P\,\mathrm{diag}(\sqrt{2}I^r,0)P^{-1}$

由于P为正交阵，故B为实对称方阵.

且有 $I+A=B^2$

8分

Ⅱ. 正交变换

对应矩阵 $A=\begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$|\lambda I-A|=\begin{vmatrix} \lambda & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \lambda & 0 \\ -1 & 0 & \lambda \end{vmatrix}=\lambda\left(\lambda^2-\frac{5}{4}\right)=0$$

(1) $(0\cdot I-A)x=0$, $\begin{pmatrix} 0 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$, $\alpha_1=(0,2,-1)^T$

(2) $(\frac{\sqrt{5}}{2}I-A)x=0$, $\begin{pmatrix} \frac{\sqrt{5}}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{\sqrt{5}}{2} & 0 \\ -1 & 0 & \frac{\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$, $\alpha_2=(\frac{\sqrt{5}}{2},\frac{1}{2},1)^T$

(3) $(-\frac{\sqrt{5}}{2}I-A)x=0$, $\begin{pmatrix} -\frac{\sqrt{5}}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{2} & -\frac{\sqrt{5}}{2} & 0 \\ -1 & 0 & -\frac{\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$, $\alpha_3=(-\frac{\sqrt{5}}{2},\frac{1}{2},1)^T$

将 $\alpha_1,\alpha_2,\alpha_3$ Schmidt 正交化，

得标准正交基 $e_1=\frac{1}{\sqrt{5}}(0,2,-1)^T$, $e_2=\frac{1}{\sqrt{10}}(\sqrt{5},1,2)^T$, $e_3=\frac{1}{\sqrt{10}}(-\sqrt{5},1,2)^T$

令 $P=\frac{1}{\sqrt{10}}\begin{pmatrix} 0 & \sqrt{5} & -\sqrt{5} \\ 2\sqrt{2} & 1 & 1 \\ -\sqrt{2} & 2 & 2 \end{pmatrix}$, 则 $P^{-1}AP=\mathrm{diag}(0,\frac{\sqrt{5}}{2},-\frac{\sqrt{5}}{2})$

则变换为 $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=P\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$