一、

1.基于$x_s, s<t$和$y_s, s<t$对y_t的预测误差比基于$y_s, s<t$对y_t预测误差小即$Var(y_t-E[y_t|y_{t-1},\dots]) \geq Var(y_t-E[y_t|y_{t-1},\dots,x_{t-1},\dots])$ $\quad N^2p+\frac{1}{2}N(N+1)$

2.$-0.5<a<1$

3.$\frac{1}{1-\phi_1/2-\cdots-\phi_p/2^p}$

4.$\begin{cases} 1+\phi_1^2+\cdots+\phi_{l-1}^2 & l\leq q \\ 1+\phi_1^2+\cdots+\phi_q^2 & l>q \end{cases}$

5.12 $\quad ARIMA(0,1,1)\times(0,1,1)_{12}$

6.GJR-$GARCH$ $(TGARCH)$ $\quad EGARCH$

7.$EX_t=\mu, EX_t^2<+\infty, EX_tX_s$ 只与$(t-s)$有关

8.$\frac{\rho_1(1-\rho_2)}{1-\rho_1^2}=\frac{8}{21}$ $\quad \frac{\rho_2-\rho_1^2}{1-\rho_1^2}=\frac{1}{21}$

9.$cos\omega(t-s)$

二、

1. p=1 d=1 q=1

$Y_t\sim ARIMA(1,1,1)$ $\quad \nabla Y_t\ ARMA(1,1)$

$\because \nabla Y_t=10+0.5\nabla Y_{t-1}+\varepsilon_t-0.5\varepsilon_{t-1}$

$\therefore E\nabla Y_t=\frac{10}{1-0.5}=20$

令$Var(\nabla Y_t)=\gamma_0$ $\quad$ 两边同乘ε_t $\quad E\nabla Y_t\varepsilon_t=E\varepsilon_t^2=1$

$\because Var(\nabla Y_t)=Var(10+0.5\nabla Y_{t-1}+\varepsilon_t-0.5\varepsilon_{t-1})$

$\therefore \gamma_0=0.25\gamma_0+1+0.25-2\times0.25E\nabla Y_{t-1}\varepsilon_{t-1}=0.25\gamma_0+0.75$

$\therefore \gamma_0=1$

2. $\quad EX_t\varepsilon_t=E\varepsilon_t^2=\sigma^2$

$\begin{cases} \gamma_1=EX_tX_{t-1}=0.5\gamma_0-0.25\sigma^2 \\ \gamma_0+0.25\gamma_0-\gamma_1=(1+0.25^2)\sigma^2 \end{cases}$

解得$\gamma_0=\frac{13}{12}\sigma^2$ $\quad \gamma_1=\frac{7}{24}\sigma^2$ $\quad \therefore \rho_1=\frac{\gamma_1}{\gamma_1}=\frac{7}{26}$

又$\because \gamma_k=0.5\gamma_{k-1}, k\geq2$ $\quad \therefore \rho_k=0.5\rho_{k-1}, k\geq2$

$\therefore \rho_k=\frac{7}{26}\times(\frac{1}{2})^{k-1}, k\geq2$

3. 易知$\rho_1=\frac{\theta}{1+\theta^2}$

$\because (1+\theta)^2=1+\theta^2+2\theta\geq0$ $\quad (1-\theta)^2=1+\theta^2-2\theta\geq0$

$\therefore 1+\theta^2\geq-2\theta$ $\quad 1+\theta^2\geq2\theta$

$\therefore |\rho_1|\leq\frac{1}{2}$

三、

1.(1) $|-0.5|<1, -0.5\pm(-1)<1$, 平稳;

$0.4<1$,可逆。

(2) 由$\psi_0=0$递推式$\psi_j=b_j+\sum_{k=1}^p a_k\psi_{j-k}, j=1,2,\dots$(其中$b_j=0, j>q$,$\psi_j=0, j<0$)知

$\psi_1=b1+\psi_0=1.4$

$\psi_2=\psi_1-0.5\psi_0=0.9$

$\psi_3=\psi_2-0.5\psi_1=0.2$

(3)

$$
\begin{aligned}
\epsilon_t &= \frac{1-B+0.5B^2}{1+0.4B}X_t \\
&= (1-B+0.5B^2)\sum_{j=0}^{\infty}(-0.4B^j)X_t \\
&= \Big(\sum_{j=0}^{\infty}(-0.4)^jB^j - \sum_{j=0}^{\infty}(-0.4)^jB^{j+1} + \sum_{j=0}^{\infty}0.5(-0.4)^jB^{j+2}\Big)X_t \\
&=
\end{aligned}
$$

2.(1) 记$u_t = y_t^2 - h_t$

$$
\begin{aligned}
h_t + y_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1} + y_t^2 \\
&= \alpha_0 + \alpha_1(y_{t-1}^2 - h_{t-1}) + (\alpha_1+\beta_1)h_{t-1} + y_t^2
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow y_t^2 &= \alpha_0 + (\alpha_1+\beta_1)(h_{t-1}+u_{t-1}) + \alpha_1(y_{t-1}^2 - h_{t-1}) - (\alpha_1+\beta_1)u_{t-1} + y_t^2 \\
&= \alpha_0 + (\alpha_1+\beta_1)y_{t-1}^2 + u_t - \beta_1 u_{t-1}
\end{aligned}
$$

下证$u_t \sim WN(0,\sigma^2)$：

$Eu_t = Ey_t^2 - Eh_t = E(h_t\epsilon_t^2) - Eh_t = Eh_tE\epsilon_t^2 - Eh_t = 0$

$$
\begin{aligned}
\because Cov(u_t,u_s) &= E(u_tu_s) - Eu_tEu_s \\
&= E(u_tu_s) \\
&= E[(y_t^2-h_t)(y_s^2-h_s)] \\
&= E[h_th_s(\epsilon_t^2-1)(\epsilon_s^2-1)]
\end{aligned}
$$

若$s<t$, $\quad Cov(u_t,u_s) = E[h_th_s(\epsilon_t^2-1)(\epsilon_s^2-1)] = 0$；

若$s=t$, $\quad Cov(u_t,u_s) = E(h_t^2)E[(\epsilon_t^2-1)^2] = 2E(h_t^2) =$常数,

故u_t为一白噪声。

$\because E(y_t^4) = 3Eh_t^2$

$$
\begin{aligned}
E(y_t^4) &= [3 + \frac{6\alpha_1^2}{1-2\alpha_1^2-(\alpha_1+\beta_1)^2}]E(y_t^2)^2 \\
&= [3 + \frac{6\alpha_1^2}{1-2\alpha_1^2-(\alpha_1+\beta_1)^2}](\frac{\alpha_0}{1-\alpha_1-\beta_1})^2 \\
&\triangleq M
\end{aligned}
$$

$\Rightarrow Eh_t^2 = \frac{M}{3}$

(2) 由于$Eh_t = \alpha_0 + \alpha_1 Ey_{t-1}^2 + \beta_1 Eh_{t-1}$

且$Ey_{t-1}^2 = Eh_{t-1}$

$\Rightarrow Ey_{t-1}^2 = \frac{\alpha_0}{1-\alpha_1-\beta_1}$

$\gamma_k = Cov(y_t^2, y_{t+k}^2) = Cov(h_t\epsilon_t^2, h_{t+k}\epsilon_{t+k}^2)$

若$k=0$, $\quad Var(y_t^2) = M - (\frac{\alpha_0}{1-\alpha_1-\beta_1})^2 = [2 + \frac{6\alpha_1^2}{1-2\alpha_1^2-(\alpha_1+\beta_1)^2}](\frac{\alpha_0}{1-\alpha_1-\beta_1})^2$

若$k \geq 1$,

$$
\begin{aligned}
\gamma_k &= Cov(y_t^2, y_{t+k}^2) \\
&= Cov(u_t, u_{t+k}) + Cov(h_t, y_{t+k}^2) + Cov(y_t^2, h_{t+k}) - Cov(h_t, h_{t+k}) \\
&= Cov(h_t, h_{t+k}\epsilon_{t+k}^2) + Cov(h_t\epsilon_t^2, h_{t+k}) - Cov(h_t, h_{t+k}) \\
&= Cov(h_t\epsilon_t^2, h_{t+k})
\end{aligned}
$$

而由于y_t^2为ARMA(1,1)

$$
\therefore \quad \gamma_k - (\alpha_1+\beta_1)\gamma_{k-1} = \begin{cases} \dfrac{2M}{3}(-\beta_1), & k=1 \\ 0, & k>1 \end{cases}
$$

故由此可递推得$\{\gamma_k\}_{k=0}^{\infty}$.

3.(1)由ARMA定义知$E(X_s, \epsilon_t) = 0, \forall s < t, \quad$ 且$\epsilon_t \sim WN(O, \sigma^2)$.

故$Cov(X_t, \epsilon_t) = Cov(\phi_1 X_{t-1} + \epsilon_t + \theta_1\epsilon_{t-1}, \epsilon_t) = \sigma^2$.

$$
\begin{aligned}
\gamma_0 &= Cov(X_t, X_t) \\
&= Cov(\phi_1 X_{t-1} + \epsilon_t + \theta_1\epsilon_{t-1}, \phi_1 X_{t-1} + \epsilon_t + \theta_1\epsilon_{t-1}) \\
&= \phi_1^2\gamma_0 + (2\phi_1\theta_1 + \theta_1^2 + 1)\sigma^2
\end{aligned}
$$

$\Rightarrow \gamma_0 = \frac{2\phi_1\theta_1+\theta_1^2+1}{1-\phi_1^2}\sigma^2$

$\gamma_1 = Cov(X_t, X_{t-1}) = \gamma_0 + \theta_1\sigma^2$

$\gamma_2 = Cov(X_t, X_{t-2}) = \phi_1\gamma_1$

且$\gamma_n = Cov(X_t, X_{t-n}) = \phi_1\gamma_{n-1} = \phi_1^{n-1}\gamma_1 \quad (\forall n \geq 2)$

故$\Gamma = \begin{pmatrix} Cov(X_t, X_t) & Cov(X_t, X_{t-1}) \\ Cov(X_t, X_{t-1}) & Cov(X_{t-1}, X_{t-1}) \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}$

$\beta = \begin{pmatrix} Cov(X_t, Y_{t+1}) \\ Cov(X_{t-1}, Y_{t+1}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\gamma_0+\gamma_1) \\ \frac{1}{2}(\gamma_1+\gamma_2) \end{pmatrix}$

$\Rightarrow a = \Gamma^{-1}\beta$

故最佳线性预测为$a(X_t, X_{t-1})^T$.

均方误差为：

$$
\begin{aligned}
Var(Y_{t+1}) - \beta^T\Gamma^{-1}\beta &= Var(\frac{1}{2}(X_{t+1}+X_t)) - \beta^T\Gamma^{-1}\beta \\
&= \frac{1}{4}Var(X_{t+1}) + \frac{1}{2}Cov(X_{t+1}, X_t) - \beta^T\Gamma^{-1}\beta \\
&= \frac{1}{4}\gamma_0 + \frac{1}{2}\gamma_1 - \beta^T\Gamma^{-1}\beta
\end{aligned} \tag{1}
$$

其中$\beta^T\Gamma^{-1}\beta$可由之前结果计算得。

(2)记$\{Y_t\}$的协方差函数为$\{\hat{\gamma_k}\}$.

则$\{\hat{\gamma_k}\} = Cov(Y_t, Y_{t+k}) = \frac{1}{4}(2\gamma_k + \gamma_{k+1} + \gamma_{k-1})$.

易知，γ_k为绝对可和的。
而

$$\sum_{k=-\infty}^{\infty}|\hat{\gamma_k}| \le \frac{1}{4}\sum_{k=-\infty}^{\infty}(2|\gamma_k|+|\gamma_{k+1}|+|\gamma_{k-1}|)$$
$$= \sum_{k=-\infty}^{\infty}|\gamma_k|$$
$$< \infty$$

故$\{Y_t\}$有谱密度：
$f(\lambda)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\hat{\gamma_k}e^{-ik\lambda}$,
$\{\hat{\gamma_k}\}$可由(1)中γ_k算得。