一、

1.cos(ω(s-t)).

2.$-1 < a < 0.9$.

3.$\frac{1}{1+\theta_1+\cdots+\theta_q}$.

4.$\frac{\sigma^2}{2\pi}\left|\frac{B(e^{i\lambda})}{A(e^{i\lambda})}\right|^2$, 其中$A(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, $B(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$.

5.12; $SARIMA(0,0,1)\times(0,1,0)_{12}$; $\frac{1+\theta^2}{1-\Phi^2}\sigma^2$.

6.白噪声; χ^2; 该序列为i.i.d的白噪声

7.差分d次平稳

8.用$\{X_t, t \le n\}$对X_{n+1} 做线性预测的误差非零; $\lim\limits_{k\to\infty}\sigma_k^2 = \gamma_0$

9.$\frac{a_1}{1-a_2}$; a_2; 0

二、

1. $\widehat{a}_1 = \frac{1-\widehat{\rho}_2}{1-\widehat{\rho}_1^2}\widehat{\rho}_1 \approx 1.435$ $\widehat{a}_2 = \frac{\widehat{\rho}_2-\widehat{\rho}_1^2}{1-\widehat{\rho}_1^2}\widehat{\rho}_1 \approx -0.721$

$\widehat{\sigma}^2 = \widehat{\gamma}_0(1 - \widehat{a}_1\widehat{\rho}_1 - \widehat{a}_2\widehat{\rho}_2) \approx 1.219$

$\because \sqrt{n}(\widehat{a}_1 - a_1, \widehat{a}_2 - a_2) \to n(0, \sigma^2\Gamma_2^{-1})$

$\therefore Cov(\vec{a}) = \frac{1}{n}\widehat{\sigma}^2\widehat{\Gamma}_2^{-1} = \frac{1}{144}\times 1.219 \times \begin{pmatrix} \widehat{\gamma}_0 & \widehat{\gamma}_1 \\ \widehat{\gamma}_1 & \widehat{\gamma}_0 \end{pmatrix}^{-1} \approx \begin{pmatrix} 0.003 & -0.002 \\ -0.002 & 0.003 \end{pmatrix}$

2. 直接累加得$X_t = tc + \epsilon_1 + \cdots + \epsilon_t + \eta_t - \eta_0 + X_0$

$\therefore Cov(X_t, X_s) = \begin{cases} t\sigma_\epsilon^2 & t < s \\ t\sigma_\epsilon^2 + \sigma_\eta^2 & t = s \end{cases}$, $E(X_t) = tc + X_0 - \eta_0$

3. $\because Cov(X_t, X_{t+k}) = Cov(\epsilon_t + \theta\epsilon_{t-12}, \epsilon_{t+k} + \theta\epsilon_{t+k-12}) = \begin{cases} (1+\theta^2)\sigma^2 & k = 0 \\ \theta\sigma^2 & k = 12 \\ 0 & k \ne 0, 12 \end{cases}$

$\therefore \rho_k = \begin{cases} 1 & k = 0 \\ \frac{\theta}{1+\theta^2} & k = 12 \\ 0 & k \ne 0, 12 \end{cases}$

三、

1.

(1)

$\widehat{Y}_{t+1} = 40 + E(\epsilon_{t+1}) - 0.6\epsilon_t + 0.8\epsilon_{t-1} = 35.6$

$\widehat{Y}_{t+2} = 40 + E(\epsilon_{t+2}) - 0.6E(\epsilon_{t+1} + 0.8\epsilon_t) = 41.6$

(2)

$\because \widehat{Y}_{t+1} \overset{d}{=} 35.6 + \epsilon_{t+1}$ $\widehat{Y}_{t+2} \overset{d}{=} 41.6 + \epsilon_{t+2} - 0.6\epsilon_{t+1}$

$\epsilon_{t+1}, \epsilon_{t+2} \sim N(0, 20)$

$\therefore \widehat{Y}_{t+1}$ 的95% CI 为$(35.6 - 1.96\sqrt{20}, 35.6 + 1.96\sqrt{20}) \approx (26.835, 44, 365)$

$\therefore \widehat{Y}_{t+2}$ 的95% CI 为$(41.6 - 1.96\times(1+0.6^2)\sqrt{20}, 41.6 + 1.96\times(1+0.6^2)\sqrt{20}) \approx (31.378, 51.822)$

(3)

$\because \gamma_0 = 40,$ $\gamma_1 = Cov(Y_t, Y_{t-1}) = -21.6,$ $\gamma_2 = Cov(Y_t, Y_{t-2}) = 16, \gamma_k = 0, (k > 2)$

$\therefore f(\lambda) = \frac{1}{2\pi}\sum_{k=-2}^{k=2}\gamma_k e^{-ik\lambda} = \frac{1}{2\pi}(40 - 43.2cos(\lambda) + 32cos(2\lambda))$

2.

(1)

$\because v_t \sim N(0,1) \qquad \epsilon_t = v_t\sqrt{1+0.05\epsilon_{t-1}^2}$

$\therefore \epsilon_t|_{\epsilon_{t-1}=\epsilon} \sim N(0, \sqrt{1+0.05\epsilon^2})$

$\therefore h_t = Var(\epsilon_t|_{\epsilon_{t-1}=\epsilon}) = 1+0.05\epsilon^2$ 是一个常数.

$\therefore E(Y_t|X_t = 0.1, \epsilon_{t-1} = 0.6) = 0.05 + 0.3\times 0.1 + 0.2\times(1+0.05\times 0.6^2) = 0.2836$

(2)

易知$Var(Y_t|X_t = 0.1, \epsilon_{t-1} = 0.6) = Var(\epsilon_t|_{\epsilon_{t-1}=0.6}) = 1 + 0.05\times 0.6^2 = 1.018$

(3)

由上面推导得$Y_t|X_t = x, \epsilon_{t-1} = \epsilon \sim N(0.25 + 0.3x + 0.01\epsilon^2, 1 + 0.05\epsilon^2)$ 服从正太分布。

3.

(1)

不妨令$s < t$

case1 $\qquad 1 \le t < s \le m$

$E(Y_tY_s) = \frac{1}{\sigma^2}E(X_tX_s) = \frac{1}{\sigma^2}\gamma_{t-s}$

记$\{\gamma_k\}$ 为$\{X_t\}$ 的自协方差函数则

当$k < q$

$\gamma_k = \sum_{j=1}^p a_j\gamma_{k-j} + \sum_{j=0}^q b_j\psi_{j-k}\sigma^2$

其中

$$\psi_j = \begin{cases} 1 & j = 0 \\ b_j + \sum_{k=1}^p a_k\psi_{j-k} & j = 1,\dots,q \\ 0 & else \end{cases}$$

当$k = q$

$\gamma_k = \sum_{j=1}^p a_j\gamma_{k-j} + \sigma^2 b_q$

当$k > q$

$\gamma_k = \sum_{j=1}^p a_j\gamma_{k-j}$

case2 $\qquad 1 \le s \le m < t$

$$E(Y_tY_s) = \frac{1}{\sigma^2}E(X_s(X_t - \textstyle\sum_{j=1}^p a_jX_{t-j})) = \begin{cases} \frac{1}{\sigma^2}(\gamma_{t-s} - \sum_{j=1}^p a_j\gamma_{t-s-j}) & t-s < q \\ 0 & t-s \ge q \end{cases}$$

case3 $\qquad m < s < t$

$$E(Y_tY_s) = \frac{1}{\sigma^2}E\left[(\textstyle\sum_{j=0}^q b_j\epsilon_{t-j})(\sum_{j=0}^q b_j\epsilon_{s-j})\right] = \begin{cases} \sum_{j=0}^q b_jb_{j+t-s} & t-s \le q \\ 0 & t-s > q \end{cases}$$

其中$b_0 = 1$

(2)

令$A(L)X_t = B(L)\epsilon_t$ 其中$A(L) = 1 - \sum_{j=1}^p a_jL^j, \qquad B(L) = 1 + \sum_{j=1}^q b_jL^j$

则$A(L)z_t = B(L)\epsilon_t + A(L)\eta_t$

$\because \{\epsilon_t\}$ 和$\{\eta_t\}$ 独立

$\therefore B(L)\epsilon_t + A(L)\eta_t$ 可以看成$MA(m)$ 序列，其中$m = max(p,q)$.

所以$\{z_t\} \sim ARMA(p,m)$