答案 2017

一、 填空

1. 2.5 0.8 0.8

2. $X_t = (1-\theta_1 B)(1-\theta_{12}B)\varepsilon_t, \varepsilon_t \sim WN(0,\sigma^2)$ $\frac{-\theta_1}{1+\theta^2{}_1}, \frac{-\theta_{12}}{1+\theta^2{}_{12}}, \frac{\theta_1\theta_{12}}{(1+\theta^2{}_{12})(1+\theta_1{}^2)}$,

3. $\frac{1+ts}{\sqrt{(1+t^2)(1+s^2)}}$

4. 5.7625

5. $\{(\phi_1,\phi_2):|\phi_2|\le 1, \phi_2 \pm \phi_1 \le 1\}, \frac{\phi_0}{1-\phi_1-\phi_2}$

6. 随机游动，$\sqrt{\frac{\min\{t,s\}}{\max\{t,s\}}}$，单位根，是个平稳过程

7. $\frac{1}{1-\phi_1-\cdots-\phi_p}$

8. $\gamma_X(0)$

二、(1)

$Y_t = 0.8Y_{t-1} + \varepsilon_t + 0.7\varepsilon_{t-1} + 0.6\varepsilon_{t-2}$,

两边都乘以Y_{t-k},再取期望

$\gamma_k = 0.8\gamma_{k-1}, \quad k \ge 3,$

$\gamma_2 = 0.8\gamma_1 + 0.6\sigma^2, \quad k = 2,$

$\gamma_1 = 0.8\gamma_0 + 0.7\sigma^2 + 0.6EY_{t-1}\varepsilon_{t-2}, k = 1.$

计算$EY_{t-1}\varepsilon_{t-2}$=$1.5\sigma^2$，代入上式有

$\gamma_1 = 0.8\gamma_0 + 2.2\sigma^2$

于是得到ρ_k的递推式

$\rho_k = 0.8\rho_{k-1}, \quad k \ge 3,$

$\rho_2 = 0.8\rho_1 + 0.6\sigma^2 / \gamma_0,$

$\rho_1 = 0.8 + 2.2\sigma^2 / \gamma_0$

注：$\rho_1 = 0.8 + 1.6\sigma^2 / \gamma_0$

（2）

$(1-0.8B)X_t=(1+0.7B+0.6B^2)\varepsilon_t$

因为$0.8<1$,所以是平稳的，又因为$0.6<1,-0.6\pm(-0.7)<1$

所以是可逆的。

（3）传递形式为

$$(1-0.8B)X_t=(1+0.7B+0.6B^2)\varepsilon_t$$

于是

$$\begin{aligned}X_t&=\frac{1+0.7B+0.6B^2}{1-0.8B}\varepsilon_t\\&=\sum_{j=0}^{\infty}0.8^jB^j(1+0.7B+0.6B^2)\varepsilon_t\\&=\sum_{j=0}^{\infty}0.8^j\varepsilon_{t-j}+0.7\sum_{j=0}^{\infty}0.8^j\varepsilon_{t-j-1}+0.6\sum_{j=0}^{\infty}0.8^j\varepsilon_{t-j-2}\end{aligned}$$

三、（1） $|\theta_1|<1,\ |\theta_2|<1$时，序列是平稳可逆的

（2）利用 Yule—Walker 方程

$$E(X_t)=\theta_0+\theta_1E(X_{t-1})+0$$
$$\Rightarrow E(X_t)=\frac{\theta_0}{1-\theta_1}$$

$$Var(X_t)=Var(\theta_0+\theta_1X_{t-1}+Z_t+\theta_2Z_{t-1})$$
$$=\theta_1^2Var(X_{t-1})+Var(Z_t)+\theta_2^2Var(Z_{t-1})+2\theta_1\theta_2Cov(X_{t-1},Z_{t-1})$$
$$=\theta_1^2Var(X_t)+1+\theta_2^2+2\theta_1\theta_2$$
$$\Rightarrow Var(X_t)=\frac{1+\theta_2^2+2\theta_1\theta_2}{1-\theta_1^2}$$

$$\gamma(1)=Cov(X_t,X_{t-1})$$
$$=Cov(\theta_0+\theta_1X_{t-1}+Z_T+\theta_2Z_{T-1},X_{t-1})$$
$$=\theta_1Var(X_t)+\theta_2Cov(Z_{t-1},X_{t-1})$$
$$=\theta_1\gamma(0)+\theta_2*1$$

So we get:

$$\rho(1)=\frac{\gamma(1)}{\gamma(0)}=\theta_1+\theta_2*\frac{1-\theta_1^2}{1+\theta_2^2+2\theta_1\theta_2}$$

$$\begin{cases} E(X_t)=\dfrac{\theta_0}{1-\theta_1}=\bar{X} \\ Var(X_t)=\dfrac{1+\theta_2^2+2\theta_1\theta_2}{1-\theta_1^2}=S^2=\dfrac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} \\ \rho(1)=\theta_1+\theta_2*\dfrac{1-\theta_1^2}{1+\theta_2^2+2\theta_1\theta_2}=\hat{\rho}(1)=\dfrac{\sum_{i=1}^{n-1}(x_i-\bar{x})(x_{i+1}-\bar{x})}{\sum_{i=1}^{n}(x_i-\bar{x})^2} \end{cases}$$

解这个方程，我们就得到$\theta_0,\theta_1,\theta_2$的矩估计

（3）

$$X_{T+1}=\theta_0+\theta_1X_T+Z_{T+1}+\theta_2Z_T=\theta_0+\theta_1X_T+\theta_2Z_T+Z_{T+1}$$
$$X_{T+2}=\theta_0+\theta_1X_{T+1}+Z_{T+2}+\theta_2Z_{T+1}=\theta_0+\theta_1(\theta_0+\theta_1X_T+Z_{T+1}+\theta_2Z_T)+Z_{T+1}+\theta_2Z_T$$
$$=\theta_0(1+\theta_1)+\theta_1(\theta_1X_T+\theta_2Z_T)+(\theta_1+\theta_2)Z_{T+1}+Z_{T+2}$$
$$X_{T+3}=\theta_0+\theta_1X_{T+2}+Z_{T+3}+\theta_2Z_{T+2}$$
$$=\theta_0(1+\theta_1+\theta_1^2)+\theta_1^3X_T+\theta_1^2\theta_2Z_T+\theta_1(\theta_1+\theta_2)Z_{T+1}+(\theta_1+\theta_2)Z_{T+2}+Z_{T+3}$$
$$=\theta_0(1+\theta_1+\theta_1^2)+\theta_1^2(\theta_1X_T+\theta_2Z_T)+(\theta_1+\theta_2)(\theta_1Z_{T+1}+Z_{T+2})+Z_{T+3}$$

类推可得

$$X_{T+k}=\theta_0\left(1+\theta_1+\theta_1^2+\cdots+\theta_1^{k-1}\right)+\theta_1^{k-1}(\theta_1X_T+\theta_2Z_T)+Z_{T+k}+(\theta_1+\theta_2)(\theta_1^{k-2}Z_{T+1}+\theta_1^{k-3}Z_{T+2}+\cdots+Z_{T+k-1})$$

于是

$$X_{T+1,T}=E(X_{T+1}|I_T)=E(\theta_0+\theta_1X_T+\theta_2Z_T+Z_{T+1}|I_T)=\theta_0+\theta_1X_T+\theta_2Z_T$$

$$X_{T+k,T}=E(X_{T+k}|I_T)$$
$$=E\left(\theta_0\left(1+\theta_1+\theta_1^2+\cdots+\theta_1^{k-1}\right)+\theta_1^{k-1}(\theta_1X_T+\theta_2Z_T)+Z_{T+k}+(\theta_1+\theta_2)\left(\theta_1^{k-2}Z_{T+1}+\theta_1^{k-3}Z_{T+2}+\cdots+Z_{T+k-1}\right.\right.$$
$$=\theta_0\left(1+\theta_1+\theta_1^2+\cdots+\theta_1^{k-1}\right)+\theta_1^{k-1}(\theta_1X_T+\theta_2Z_T)$$
$$=\frac{\theta_0\left(1-\theta_1^k\right)}{1-\theta_1}+\theta_1^{k-1}(\theta_1X_T+\theta_2Z_T)$$

Based on the direct deviation in part (c), we can also easily derive the forecast error as:

$$e_{T+1,T} = X_{T+1} - X_{T+1,T} = Z_{T+1}$$

In general, for $k \geq 2$, we have:

$e_{T+K,T} = X_{T+k} - X_{T+k,T}$

$= Z_{T+k} + (\theta_1 + \theta_2)\left(\theta_1^{k-2} Z_{T+1} + \theta_1^{k-3} Z_{T+2} + \cdots + Z_{T+k-1}\right)$

$= Z_{T+k} + (\theta_1 + \theta_2) \sum_{i=0}^{k-2} \theta_1^i Z_{T+k-i-1}$

Therefore,

The mean of $e_{T+k,T}$ is

$$E\left[e_{T+k,T}\right] = E[Z_{T+1}] = 0$$

For $k \geq 2$, we have:

$$E\left[e_{T+k,T}\right] = E\left[Z_{T+k} + (\theta_1 + \theta_2) \sum_{i=0}^{k-2} \theta_1^i Z_{T+k-i-1}\right] = 0$$

The variance of $e_{T+k,T}$ is

$$Var\left[e_{T+k,T}\right] = Var\left[Z_{T+k} + (\theta_1 + \theta_2) \sum_{i=0}^{k-2} \theta_1^i Z_{T+k-i-1}\right]$$

$$= Var[Z_{T+k}] + (\theta_1 + \theta_2)^2 \sum_{i=0}^{k-2} \theta_1^{2i} Var[Z_{T+k-i-1}]$$

$$= 1 + (\theta_1 + \theta_2)^2 \frac{1 - \theta_1^{2(k-1)}}{1 - \theta_1^2}$$

四、

观察数据认为 PACF 二阶截尾，ACF 拖尾，故考虑 ARMA（2，0）模型拟合。

均值为零，模型即为$X_T - \beta_1 X_{T-1} - \beta_2 X_{T-2} = \varepsilon_T$，$\varepsilon_T \sim WN(0，\sigma^2)$

由数据$\hat{\rho}_1 = 0.340$，$\hat{\rho}_2 = 0.321$

故由 Y-W 方程有

$$\hat{\beta}_1 = \frac{1 - \hat{\rho}_2}{1 - \hat{\rho}_1^{\,2}} \hat{\rho}_1 = 0.261;\quad \hat{\beta}_2 = \frac{\hat{\rho}_2 - \hat{\rho}_1^{\,2}}{1 - \hat{\rho}_1^{\,2}} = 0.232$$

$$\hat{\sigma}^2 = \hat{\gamma}_0\left(1 - \hat{\beta}_1 \hat{\rho}_1 - \hat{\beta}_2 \hat{\rho}_2\right) = 2.508$$

五、

Now

$$\Gamma = \begin{pmatrix} \mathrm{Cov}(X_t, X_t) & \mathrm{Cov}(X_t, X_{t-1}) \\ \mathrm{Cov}(X_{t-1}, X_t) & \mathrm{Cov}(X_{t-1}, X_{t-1}) \end{pmatrix} = \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix} = \begin{pmatrix} 1.52 & -0.64 \\ -0.64 & 1.52 \end{pmatrix} \sigma^2,$$

and

$$\begin{aligned} \beta &= \begin{pmatrix} \mathrm{Cov}(X_t, U) \\ \mathrm{Cov}(X_{t-1}, U) \end{pmatrix} = \begin{pmatrix} \mathrm{Cov}(X_t, X_{t+1} + X_{t+2}) \\ \mathrm{Cov}(X_{t-1}, X_{t+1} + X_{t+2}) \end{pmatrix} \\ &= \begin{pmatrix} \gamma(1) + \gamma(2) \\ \gamma(2) + \gamma(3) \end{pmatrix} = \begin{pmatrix} -0.04 \\ 0.6 \end{pmatrix} \sigma^2. \end{aligned}$$

$$\boldsymbol{a} = \Gamma^{-1}\boldsymbol{\beta} = \frac{1}{1.52^2 - 0.64^2} \begin{pmatrix} 1.52 & 0.64 \\ 0.64 & 1.52 \end{pmatrix} \begin{pmatrix} -0.04 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.1700 \\ 0.4663 \end{pmatrix}$$

于是此时的最佳预测 为

$$0.1700X_t + 0.4663X_{t-1}$$

误差为

$$\begin{aligned} \mathrm{Var}(U) - \boldsymbol{\beta}^T\Gamma^{-1}\boldsymbol{\beta} &= \mathrm{Var}(X_{t+1} + X_{t+2}) - (-0.04\ 0.6) \begin{pmatrix} 0.1700 \\ 0.4663 \end{pmatrix} \sigma^2 \\ &= \mathrm{Var}(X_{t+1}) + \mathrm{Var}(X_{t+2}) + 2\,\mathrm{Cov}(X_{t+1}, X_{t+2}) - 0.2730\sigma^2 \\ &= 2\gamma(0) + 2\gamma(1) - 0.2730\sigma^2 = 1.76\sigma^2 - 0.2730\sigma^2 = 1.4870\sigma^2 = 4.461. \end{aligned}$$

（2）-0.6+0.4<1,-0.6-0.4<1，所以可逆

逆转形式把两个虚根解出来，$\lambda_1, \lambda_1 = \dfrac{0.4 \pm \sqrt{-0.08}}{2}$

$$X_t = (1 - \lambda_1 B)(1 - \lambda_2 B)\varepsilon_t$$

$$\varepsilon_t = \frac{1}{(1 - \lambda_1 B)(1 - \lambda_2 B)} X_t$$

$$= \left(\frac{\lambda_2 / (\lambda_2 - \lambda_1)}{1 - \lambda_1 B} - \frac{\lambda_1 / (\lambda_2 - \lambda_1)}{1 - \lambda_2 B}\right) X_t$$

$$= \frac{\lambda_2}{\lambda_2 - \lambda_1} \sum_{k=0}^{\infty} \lambda_1^k X_{t-j} - \frac{\lambda_1}{\lambda_2 - \lambda_1} \sum_{k=0}^{\infty} \lambda_2^k X_{t-j}$$

（3）基于所有历史信息

$$\begin{aligned}X_{t+2}+X_{t+1} &= Z_{t+2}-0.4Z_{t+1}+0.6Z_t+Z_{t+1}-0.4Z_t+0.6Z_{t-1}\\ &= (Z_{t+2}+0.6Z_{t+1})+0.2Z_t+0.6Z_{t-1}.\end{aligned}$$

于是

$$\mathcal{P}_t(X_{t+2}+X_{t+1})=0.2Z_t+0.6Z_{t-1},$$

预测方差为

$$\sigma^2+0.6^2\sigma^2=1.36\sigma^2=4.08$$