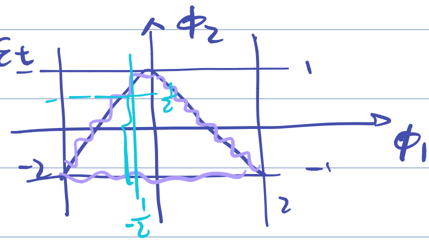
1 时间序列x_t是白噪声序列, 则$\gamma(t,s)=$ $\begin{cases}\sigma^2 & t=s\\ 0 & t\neq s\end{cases}$.

-1,1/2吧

2 设ARMA(1, 2): $X_t = 0.1X_{t-1} + \epsilon_t - 0.5\epsilon_{t-1} + a\,\epsilon_{t-2}$, 当$a$满足 $-2<a<\frac{1}{2}$ 条件时, 模型是可逆的.

$(1-0.1B)X_t=(1-0.5B+aB^2)\varepsilon_t$

平稳　　　可逆



3 设$AR(p)$模型

$$X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

的传递形式为$X_t = \mu + \underbrace{\sum_{k=0}^{\infty}\psi_k}_{Q(B)}\epsilon_{t-k}$, 则$\sum_{k=0}^{\infty}\psi_k(\frac{1}{2})^k =$ $\dfrac{1}{\sum_{k=1}^{p}\phi_k\cdot(\frac{1}{2})^k}$.

$\underbrace{(1-\phi_1 B-\cdots-\phi_p B^p)}_{P(B)}X_t=\phi_0+\varepsilon_t$

$P(B)X_t=\phi_0+\varepsilon_t$

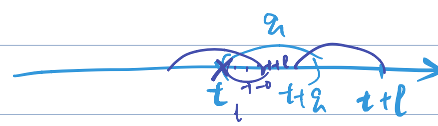
$X_t=\dfrac{\phi_0+\varepsilon_t}{P(B)}$ $\Rightarrow \dfrac{1}{P(B)}=Q(B)$. 则 $Q(\frac{1}{2})=\dfrac{1}{P(\frac{1}{2})}=\sum_{k=1}^{p}\phi_k(\frac{1}{2})^k$

4 设$\{X_t, t=0,\pm1,\pm2,\dots\}$是满足MA(q)模型

$$X_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_q\epsilon_{t-q}, \quad \epsilon_t \sim WN(0,\sigma^2)$$

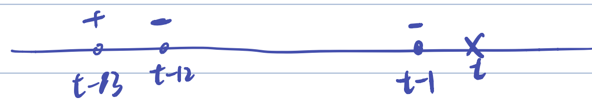
的序列, 则已知$X_t, X_{t-1},\dots$时, X_{t+l}的最佳线性预测$\hat{X}(t+l)(l\geq 1)$的均方误差是 $\|X_{t+l}-\hat{X}_t(l)\|^2=\begin{cases}(1+\theta_1^2+\cdots+\theta_q^2)\sigma^2 & l>q\\ (1+\theta_1^2+\cdots+\theta_{l-1}^2)\sigma^2 & 1\leq l\leq q\end{cases}$

$\hat{X}_t(l)=\begin{cases}\mu & l>q\\ \mu+\sum_{k=l}^{q}\theta_k\varepsilon_{t+l-k} & l\leq q\end{cases}$



$\|X_{t+l}-\hat{X}_t(l)\|^2=\begin{cases}E\left(\varepsilon_{t+l}+\theta_1\varepsilon_{t+l-1}+\cdots+\theta_q\varepsilon_{t+l-q}\right)^2=(1+\theta_1^2+\cdots+\theta_q^2)\sigma^2 & l>q\\ E\left(\sum_{k=0}^{l-1}\theta_k\varepsilon_{t+l-k}\right)^2=(1+\theta_1^2+\cdots+\theta_{l-1}^2)\sigma^2 & l\leq q\end{cases}$

5 若Y_t满足$\nabla_{12}\nabla Y_t = \epsilon_t - \theta\,\epsilon_{t-1} - \Theta\,\epsilon_{t-12} + \theta\Theta\,\epsilon_{t-13}$，该模型为一个季节周期为 12 的乘法季节模型，记为 ARIMA(0,1,1)×(0,1,1)$_{12}$



$(1-B^{12})(1-B) = \nabla_{12}\nabla Y_t = (1-\theta B)\varepsilon_t - \Theta(1-\theta B)\varepsilon_{t-12}$

$= (1-\theta B)(1-\Theta B^{12})\varepsilon_t$

6 如果模型中存在信息影响不对称现象，即好消息和坏消息对波动有不同影响．这种情况一般采用 EGARCH 模型和 NGARCH 模型．

?

不知道．

E&TGARCH？，课件里有

7 时间序列$\{X_t\}$，满足下列三个条件 ① $\forall t\in\mathbb{N}$．$EX_t^2<\infty$ 时，称其为平稳过程．

② $\forall t\in\mathbb{N}$．$EX_t=\mu$

③ $\forall t, s\in\mathbb{N}$．$E[(X_t-\mu)(X_s-\mu)]=\gamma_{t-s}$

8 $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t$，已知$\rho_1 = 0.4$，$\rho_2 = 0.2$，则$\phi_1 =$ ______，$\phi_2 =$ ______．

?

$p=2 \quad q=0$

$(1-\phi_1 B-\phi_2 B^2)Y_t = \varepsilon_t \Rightarrow Y_t = \dfrac{1}{1-\phi_1 B-\phi_2 B^2}\varepsilon_t \qquad \gamma_k = \sigma^2\sum_{j=0}^{\infty}\psi_j\psi_{j+k}$

$A(z) = 1-\phi_1 z-\phi_2 z^2 \quad B(z)=1$

$\psi(z) = \dfrac{B(z)}{A(z)} = \dfrac{1}{1-\phi_1 z-\phi_2 z^2}$

附

$\rho_1 = \dfrac{\gamma_1}{\gamma_0} = 0.4$

$\rho_2 = \dfrac{\gamma_2}{\gamma_0} = 0.2$

Wold系数递推公式：

$$\psi_j = \begin{cases} 1 & j=0 \\ b_j + \sum_{k=1}^{p} a_k \psi_{j-k} & j=1,2\cdots \end{cases}$$

$$\psi_0 = 1. \quad \psi_1 = b_1 + \phi_1 \psi_0 + \phi_2 \psi_{-1} = \phi_1$$

$$\psi_2 = b_2 + \phi_1 \psi_1 + \phi_2 \psi_0 = \phi_1^2 + \phi_2$$

$$\psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1 = \phi_1^3 + 2\phi_1\phi_2$$

$$\psi_4 = \phi_1 \psi_3 + \phi_2 \psi_2 = \phi_1^4 + 2\phi_1^2\phi_2 + \phi_1^2\phi_2 + \phi_2^2$$

$$\psi_n = \phi_1 \psi_{n-1} + \phi_2 \psi_{n-2}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\sigma^2 \cdot \sum_{j=0}^{\infty} \psi_j \psi_{j+k}}{\sigma^2 \sum_{j=0}^{\infty} \psi_j \cdot \psi_j}$$

$$\psi_{j+2} = \phi_1 \psi_{j+1} + \phi_2 \psi_j$$

$$\Rightarrow \rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0$$

代入 $\begin{cases} \rho_0 = 1 \\ \rho_1 = 0.4 \\ \rho_2 = 0.2 \end{cases}$

$$\Downarrow$$

$$0.2 = 0.4\phi_1 + \phi_2$$

然后呢？

二、

1 对ARIMA(p,d,q)模型，确定p,d,q，并求出$E\Delta Y_t$和$Var(\Delta Y_t)$.

$$Y_t = 10 + 1.5Y_{t-1} - 0.5Y_{t-2} + \epsilon_t - 0.5\epsilon_{t-1}, \quad \epsilon_t \sim WN(0,1).$$

$(1-1.5B+0.5B^2)Y_t = 10 + (1-0.5B)\varepsilon_t \Rightarrow p=2,\ q=1$

AR部分特征根：$\lambda^2 - 1.5\lambda + 0.5 = 0 \quad \lambda_1 = 1,\ \lambda_2 = 0.5$

∵ 单位特征根重数为1重 ∴ $d=1$

∴ ARIMA(2,1,1) 模型

做一阶差分：$\Delta Y_t = 10 + 1.5\Delta Y_{t-1} - 0.5\Delta Y_{t-2} + \varepsilon_t - 0.5\varepsilon_{t-1}$

直接把原式中 Y_t 换为 ΔY_t

$(1-1.5B+0.5B^2)\Delta Y_t = 10 + (1-0.5B)\varepsilon_t$

[illegible]

2 对ARMA(1,1)序列$X_t = 0.5X_{t-1} + \epsilon_t - 0.25\epsilon_{t-1}, \epsilon_t \sim WN(0, \sigma^2)$, 求解它的自相关系数$\rho_k, k \geq 2$的递推式 ?

ARMA(1,1) $\quad (1-0.5B)X_t = (1-0.25B)\varepsilon_t$

$$X_t = \frac{1-0.25B}{1-0.5B}\varepsilon_t = (1-0.25B)\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^k B^k \varepsilon_t$$

$$= \left(\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^k B^k - \sum_{k=1}^{+\infty}\left(\frac{1}{2}\right)^{k+1} B^k\right)\varepsilon_t$$

$$= \left(1 + \sum_{k=1}^{+\infty}\left(\frac{1}{2}\right)^{k+1} B^k\right)\varepsilon_t$$

$\gamma_k = \sigma^2 \cdot \sum_{j=0}^{+\infty} \Psi_j \Psi_{j+k}$ 其中 $\Psi_0 = 1$ $\Psi_k = \left(\frac{1}{2}\right)^{k+1}$ $k=1,2\cdots$

$\gamma_0 = \sigma^2 \cdot \sum_{j=0}^{+\infty} \Psi_j^2$

$\Rightarrow \rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\sum_{j=0}^{+\infty}\Psi_j\Psi_{j+k}}{\sum_{j=0}^{+\infty}\Psi_j^2}$ $\quad \because \Psi_{j+(k+1)} = \frac{1}{2}\Psi_{j+k}$ $\quad k \geq 2$ $\forall j$

$\rho_{k+1} = \frac{\sum_{j=0}^{+\infty}\Psi_j\Psi_{j+(k+1)}}{\sum_{j=0}^{+\infty}\Psi_j^2}$ $\quad \therefore \rho_{k+1} = \frac{1}{2}\rho_k \quad (k \geq 2)$

3 对任意一个MA(1)序列,

$$X_t = \epsilon_t + \theta\epsilon_{t-1}, \quad \epsilon_t \sim WN(0, \sigma^2),$$

求证它的1阶自相关系数满足$-1/2 \leq \rho_1 \leq 1/2$

$X_t = (1+\theta B)\varepsilon_t$

$$\gamma_k = \begin{cases} \sigma^2(1+\theta^2) & k=0 \\ \sigma^2\theta & k=1 \\ 0 & k \geq 2 \end{cases}$$

$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta}{1+\theta^2} = \frac{1}{\theta + \frac{1}{\theta}}$

$\because \theta + \frac{1}{\theta} \in (-\infty, -2] \cup [2, +\infty)$

$\therefore \rho_1 = \frac{\theta}{1+\theta^2} \in [-\frac{1}{2}, \frac{1}{2}]$

加上$\theta=0$的情况.

三.

1. 考虑如下的时间序列模型ARMA(2,1) $p=2$

$$(1-B+0.5B^2)X_t=(1+0.4B)\epsilon_t,\ \epsilon_t\sim WN(0,\sigma^2),$$

$a_1=1$ $a_2=-0.5$ $b_1=0.4$

(1) 判断ARMA(2,1)模型的平稳性和可逆性.

(2) 如果是平稳的, 计算线性过程$X_t=\sum_{j=0}^{\infty}\psi_j\epsilon_{t-j}$ 的系数ψ_1,ψ_2,ψ_3.

(3) 如果是可逆的, 请写出该过程的逆转形式.

(1) $\lambda^2-\lambda+0.5=0$

$\Delta=1-4\times0.5=-1$

$\lambda_{1,2}=\frac{1\pm\sqrt{\Delta}}{2}=\frac{1\pm i}{2}$

$\because |\lambda_1|=|\lambda_2|=\frac{\sqrt{2}}{2}<1$

特征根在单位圆内

$\therefore$ 平稳

$\lambda-0.4=0$

$\lambda=0.4<1$

$\therefore$ 可逆

(2) $X_t=\frac{1+0.4B}{1-B+0.5B^2}\varepsilon_t$

Wold系数递推公式:

$$\psi_j=\begin{cases}1 & j=0\\ b_j+\sum_{k=1}^{p}a_k\psi_{j-k} & j=1,2\cdots\end{cases}$$

$\psi_0=1$

$\psi_1=b_1+1\psi_0-0.5\psi_{-1}=0.4+1-0=1.4$

$\psi_2=b_2+1\psi_1-0.5\psi_0=0+1.4-0.5=0.9$

$\psi_3=b_3+1\psi_2-0.5\psi_1=0+0.9-0.5\times1.4=0.2$

(3) 逆转形式:

$$\varepsilon_t=\frac{1-B+0.5B^2}{1-(-0.4)B}X_t=\left(1-B+\frac{1}{2}B^2\right)\sum_{k=0}^{+\infty}(-0.4)^kB^kX_t$$

$$=\left(\sum_{k=0}^{+\infty}(-0.4)^kB^k-\sum_{k=0}^{+\infty}(-0.4)^kB^{k+1}+\sum_{k=0}^{+\infty}\frac{1}{2}(-0.4)^kB^{k+2}\right)X_t$$

$$=\left(1-1.4B+\sum_{k=2}^{+\infty}\left((-0.4)^k-(-0.4)^{k-1}+\frac{1}{2}(-0.4)^{k-2}\right)B^k\right)X_t$$

$$=\left(1-1.4B+\sum_{k=2}^{+\infty}1.06(-0.4)^{k-2}B^k\right)X_t$$

2. 考虑一GARCH(1,1)模型,

$$y_t = \sqrt{h_t}\epsilon_t, \epsilon_t \sim IID\ N(0,1),$$

即 σ_t^2 $$h_t = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1}, \quad \alpha_0, \alpha_1, \beta_1 \geq 0, \alpha_1 + \beta_1 < 1.$$

(1) 验证y_t^2为ARMA(1,1)模型,

$+y_t^2 - h_t \quad -\beta_1 y_{t-1}^2 + \beta_1 h_t$

$$y_t^2 = \alpha_0 + (\alpha_1 + \cancel{\beta_1}) y_{t-1}^2 + \cancel{u_t - \beta_1 u_{t-1}}, \quad u_t = y_t^2 - h_t.$$

$\alpha_0 + y_t^2 + \alpha_1 y_{t-1}^2 + (\beta_1 - 1) h_t$

(2) 计算y_t^2的均值和自协方差函数γ_k.

$\alpha_0 + h_t \varepsilon_t^2 + \alpha_1 h_{t-1} \varepsilon_{t-1}^2$

$y_t^2 = h_t \cdot \varepsilon_t^2$

$y_t^2 = (\alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1}) \varepsilon_t^2$

记 $y_t^2 = X_t$. $u_t = y_t^2 - h_t = X_t - h_t$

$\Rightarrow X_t = \alpha_0 + \alpha_1 X_t + \beta_1 h_{t-1}$

$y_t^2 = h_t \varepsilon_t^2$

$y_{t-1}^2 = h_{t-1} \varepsilon_{t-1}^2$

3. 设X_t为一ARMA(1,1)序列

$$X_t = \phi_1 X_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1}, \quad \epsilon_t \sim WN(0,\ \sigma^2).$$

$X_{t-1} = \phi_1 X_{t-2} + \varepsilon_{t-1} + \theta_1 \varepsilon_{t-2}$

令$Y_t = \frac{1}{2}(X_t + X_{t-1})$,

?

(1) 基于X_{t-1}, X_t对Y_{t+1}做最佳线性预测, 给出表达式, 并计算均方误差.

(2) 求Y_t的谱密度.

$$Y_t = \frac{1}{2}(X_t + X_{t-1}) = \frac{1}{2}(\phi_1 X_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \phi_1 X_{t-2} + \varepsilon_{t-1} + \theta_1 \varepsilon_{t-2})$$

$$= \frac{1}{2}(\phi_1 X_{t-1} + \phi_1 X_{t-2} + \varepsilon_t + (1+\theta_1)\varepsilon_{t-1} + \theta_1 \varepsilon_{t-2}) = \phi_1 Y_{t-1} + (1 + (1+\theta_1)B + \theta_1 B^2)\varepsilon_t$$

$$L[Y_{t+1} \mid X_t, X_{t-1}] = L[\tfrac{1}{2}(X_{t+1} + X_t) \mid X_t, X_{t-1}]$$

$$= L[\tfrac{1}{2}(\phi_1 X_t + \varepsilon_{t+1} + \theta_1 \varepsilon_t + X_t) \mid X_t, X_{t-1}]$$

$$= E[\tfrac{1}{2}(\phi_1 X_t + \varepsilon_{t+1} + \theta_1 \varepsilon_t + X_t) \mid X_t, X_{t-1}] = \frac{1}{2}(\phi_1 + 1) X_t$$

~~注意到,~~ $\cancel{\varepsilon_t = X_t - \phi_1 X_{t-1} - \theta_1 \varepsilon_{t-1}}$.

代入, ~~原式~~ $\cancel{= E[\tfrac{1}{2}(\phi_1 X_t + \varepsilon_{t+1} + \theta_1 \varepsilon_t + X_t)} \mid X_t, X_{t-1}]$

$$= \cancel{E[\tfrac{1}{2}(\phi_1 X_t + \varepsilon_{t+1} + \theta_1 X_t - \theta_1 \phi_1 X_{t-1} - \theta_1^2 \varepsilon_{t-1} + X_t)} \mid X_t, X_{t-1}]$$

MA-1?

$$= \cancel{\tfrac{1}{2}(\phi_1 + \theta_1 + 1) X_t - \tfrac{1}{2}\theta_1 \phi_1 X_{t-1} + E[\tfrac{1}{2}(\varepsilon_{t+1} - \theta_1^2 \varepsilon_{t-1})} \mid X_t, X_{t-1}]$$

?

$$= \cancel{\tfrac{1}{2}(\phi_1 + \theta_1 + 1) X_t - \tfrac{1}{2}\theta_1 \phi_1 X_{t-1}}$$

$$\| Y_{t+1} - \hat{Y}_{t+1} \|^2 = E\left(\tfrac{1}{2}(X_{t+1} + X_t) - \tfrac{1}{2}(\phi_1 + 1) X_t\right)^2$$

$$= E\left(\tfrac{1}{2}(X_{t+1} - \phi_1 X_t)\right)^2$$

$$= \frac{1}{4} E(\varepsilon_{t+1} + \theta_1 \varepsilon_t)^2$$

$$= \frac{1}{4}(E\varepsilon_{t+1}^2 + \theta_1^2 E\varepsilon_t^2 + 2\theta_1 E\varepsilon_t \varepsilon_{t+1})$$

$$= \frac{1}{4}\sigma^2(1 + \theta_1^2)$$

(2) Y_t 谱密度：

$$Y_t = \frac{1}{2}(X_t + X_{t-1})$$

$$\Rightarrow X_t = 2Y_t - X_{t-1}$$

$$= 2Y_t - (2Y_{t-1} - X_{t-2}) = 2Y_t - 2Y_{t-1} + X_{t-2}$$

$$= 2Y_t - 2Y_{t-1} + (2Y_{t-2} - X_{t-3}) = 2Y_t - 2Y_{t-1} + 2Y_{t-2}$$

$$\Delta Y_t = Y_t - Y_{t-1} = \frac{1}{2}(X_t - X_{t-2}) = \frac{1}{2}(\phi_1 X_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1} - \phi_1 X_{t-3} - \varepsilon_{t-2} - \theta_1\varepsilon_{t-3})$$

$$= \frac{1}{2}(\phi_1(X_{t-1} - X_{t-3}) + (1-B^2)(1+\theta_1 B)\varepsilon_t)$$

$$= \phi_1 \Delta Y_{t-1} + (1-B^2)(1+\theta_1 B)\varepsilon_t$$

$$\Rightarrow \Delta Y_t = \phi_1 \Delta Y_{t-1} + (1-B^2)(1+\theta_1 B)\varepsilon_t$$

$$Y_t = \phi_1 Y_{t-1} + (1 + (1+\theta_1)B + \theta_1 B^2)\varepsilon_t$$

$$(1-\phi_1 B)Y_t = (1+B)(1+\theta_1 B)\varepsilon_t \qquad (*)$$

由于：$(1-\phi_1 B)X_t = (1+\theta_1 B)\varepsilon_t$ 是ARMA(1,1)

$\therefore$ $1-\phi_1 z \neq 0 \quad |z| \leq 1$，$1+\theta_1 z \neq 0 \quad |z| < 1$

记 $A(z) = 1-\phi_1 z$ $\quad B(z) = (1+z)(1+\theta_1 z)$

$\therefore$ $A(z) \neq 0 \quad |z| \leq 1$. $\quad B(z) \neq 0 \quad |z| < 1$

$\therefore$ $(*)$ 是ARMA(1,2)，代入谱密度公式得：

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left|\frac{B(e^{i\lambda})}{A(e^{i\lambda})}\right|^2$$