装订线●答题时不要超过此线

中 国 科 学 技 术 大 学
2016 - 2017学年第一学期期终考试试卷(A)

考试科目: 线性代数(B1) 得分:______

学生所在院系:______ 姓名:______ 学号:______

一、【25分】填空题:

(1) 若线性方程组

$$\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 1, \\ x_1 - x_2 + x_3 + x_4 = 2, \\ 3x_1 - 4x_2 + 3x_3 + x_4 = t, \end{cases}$$

有解, 则参数$t =$ 5 .

(2) 若向量$\alpha_1 = (1,1,1)$, $\alpha_2 = (1,2,3)$, $\alpha_3 = (2,3,t)$生成$\mathbf{R}^3$中的2维子空间, 则参数$t =$ 4 .

(3) 设$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, 则$\boldsymbol{A}$的相合规范型为 $\begin{pmatrix} 5 & & \\ & -1 & \\ & & -1 \end{pmatrix}$.

(4) 二次曲面方程$2x^2-3y^2-3z^2-2yz-5=0$表示的曲面类型是 双叶双曲面 .

(5) 实二次型$Q(x,y,z) = x^2 + y^2 + 3z^2 + 2txy + 2txz + 2yz$为正定当且仅当参数$t$满足 $-1 < t < 1$.

二、【25分】判断下列命题是否正确，并简要说明理由.

(1) 设$\boldsymbol{A}$是一个n阶方阵，则$\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^2)$.

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(2) 若0是矩阵$\boldsymbol{A}$的特征值，则$\boldsymbol{A}$一定是奇异矩阵.

√

(3) 设$\boldsymbol{A}$是一个n阶方阵. 若对任意n维列向量$\boldsymbol{x}$都有$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$. 则$\boldsymbol{A}$为反对称方阵.

√

(4) 若$\boldsymbol{A}, \boldsymbol{B}$是$n$阶正定矩阵，则$\boldsymbol{AB}$也是$n$阶正定矩阵.

×

(5) 设$\boldsymbol{A}$是一个n阶方阵. 则$\boldsymbol{A}$是正交矩阵当且仅当$\boldsymbol{A}$的n个行向量组成n维实数组空间$\mathbf{R}^n$的标准正交基.

√

三、【14分】在$\mathbf{R}^3$中定义线性变换$\mathscr{A}(x,y,z)=(x+2y,x-3z,2y-z)$.

(1) 求$\mathscr{A}$在基$\alpha_1=(1,0,0)$, $\alpha_2=(1,1,0)$, $\alpha_3=(1,1,1)$下的表示矩阵;

(2) 是否存在基β_1,β_2,β_3使得$\mathscr{A}$在该基下的表示矩阵为

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}$$

解：(1) $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y \\ x-3z \\ 2y-z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + z\begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$

$$A\alpha_1 = A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$A\alpha_2 = A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$A\alpha_3 = A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow (A\alpha_1, A\alpha_2, A\alpha_3) = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= (\alpha_1,\alpha_2,\alpha_3)\cdot(\alpha_1,\alpha_2,\alpha_3)^{-1}\begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\left(\alpha_1,\alpha_2,\alpha_3 \left| \begin{matrix} 1 & 3 & 3 \\ 1 & 1 & -2 \\ 0 & 2 & 1 \end{matrix}\right.\right) = \left(\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} 1 & 3 & 3 \\ 1 & 1 & -2 \\ 0 & 2 & 1 \end{matrix}\right.\right)$$

$$\to \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \ \boxed{\begin{matrix} 0 & 2 & 5 \\ 1 & -1 & -3 \\ 0 & 2 & 1 \end{matrix}}\right)$$

所以 $(A\alpha_1, A\alpha_2, A\alpha_3) = (\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix} 0 & 2 & 5 \\ 1 & -1 & -3 \\ 0 & 2 & 1 \end{pmatrix}$.

(2) $\operatorname{Tr}\begin{pmatrix} 0 & 2 & 5 \\ 1 & -1 & -3 \\ 0 & 2 & 1 \end{pmatrix} = 0 \neq \operatorname{Tr}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix} = 3$.

不存在 β_1,β_2,β_3.

四、【14分】设$V=\{f(x)=a_0+a_1x+a_2x^2\}$为次数不超过2的实系数多项式构成的线性空间.

(1) 证明：$(f(x),g(x))=f(0)g(0)+f(1)g(1)+f(2)g(2)$ 定义了V上的一个内积.

(2) 应用Schmidt正交化方法将向量组$\{1,x\}$改造成相对于(1)中所定义内积的标准正交向量组.

(1) (f,g) 对称，线性显然.

$(f,f)=f^2(0)+f^2(1)+f^2(2)\geq 0$

若 $(f,f)=0$，$\Rightarrow f(0)=f(1)=f(2)=0$

因 f 为二次多项式. $\Rightarrow f\equiv 0$.

(2) $\alpha_1=1$，$\beta_1=\alpha_1$，$e_1=\dfrac{\beta_1}{|\beta_1|}$

$\alpha_2=x$　$\beta_2=\alpha_2-(\alpha_2,e_1)e_1$，$e_2=\dfrac{\beta_2}{|\beta_2|}$

$=\alpha_2-\dfrac{(\alpha_2,\alpha_1)\alpha_1}{|\alpha_1|^2}$

$(\alpha_1,\alpha_1)=(\beta_1,\beta_1)=(1,1)=1\cdot 1+1\cdot 1+1\cdot 1=3\Rightarrow|\beta_1|=\sqrt{3}$.

$\Rightarrow e_1=\dfrac{1}{\sqrt{3}}$

$(\alpha_1,\alpha_2)=(1,x)=1\cdot 0+1\cdot 1+1\cdot 2=3$

$\Rightarrow \beta_2=x-\dfrac{3\cdot 1}{3}=x-1$.

$(\beta_2,\beta_2)=(-1)\cdot(-1)+0\cdot 0+(2-1)(2-1)=2$.

$\Rightarrow|\beta_2|=\sqrt{2}$.

$\Rightarrow e_2=\dfrac{1}{\sqrt{2}}(x-1)$.

$\{1,x\}\longrightarrow\left\{\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{2}}(x-1)\right\}$.

五、【14分】设M是$2n$阶方阵

$$M = \begin{pmatrix} I & A \\ A & I \end{pmatrix},$$

其中A是一个n阶实方阵且满足$A^T = A$, $A^2 = I$.

(1) 求矩阵M的所有特征值;

(2) 求可逆矩阵P使得$P^{-1}MP$为对角矩阵.

解 (1) $\lambda \begin{pmatrix} I & \\ & I \end{pmatrix} - M = \begin{pmatrix} (\lambda-1)I & -A \\ -A & (\lambda-1)I \end{pmatrix}$

$$\xrightarrow{r_1 \times \left(\frac{1}{\lambda-1}A\right)} \begin{pmatrix} (\lambda-1)I, & -A \\ 0, & \frac{1}{\lambda-1}A\cdot(-A)+(\lambda-1)I \end{pmatrix}$$

$$= \begin{pmatrix} (\lambda-1)I & -A \\ 0, & \left((\lambda-1)-\frac{1}{\lambda-1}\right)I \end{pmatrix}$$

(因为 $A^2 = I$)

$$= \begin{pmatrix} (\lambda-1)I & -A \\ 0, & \frac{\lambda(\lambda-2)}{\lambda-1}I \end{pmatrix}$$

$$\Rightarrow \det\left(\lambda\begin{pmatrix} I & \\ & I \end{pmatrix} - M\right) = \det\begin{pmatrix} (\lambda-1)I & A \\ 0, & \frac{\lambda(\lambda-2)}{(\lambda-1)}I \end{pmatrix}$$

$$= \lambda^n(\lambda-2)^n.$$

$$\Rightarrow \lambda = 0 \text{ 或 } \lambda = 2.$$

(2) $\lambda = 0$, $\lambda\begin{pmatrix} I & \\ & I \end{pmatrix} - M = \begin{pmatrix} -I, & -A \\ -A, & -I \end{pmatrix} \rightarrow \begin{pmatrix} -I & -A \\ 0 & 0 \end{pmatrix}$

$\rightarrow (I \ \ A)$

$(I, A)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow x + Ay = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -Ay \\ y \end{pmatrix}$

$\Rightarrow V_0(M) = \left\{ \begin{pmatrix} -Ae_i \\ e_i \end{pmatrix} \middle| \ i = 1, 2, \cdots n \right\}$

类似地. $V_2(M) = \left\{ \begin{pmatrix} Ae_i \\ e_i \end{pmatrix} \middle| \ i = 1, 2, \cdots n \right\}$

令. $P = \begin{pmatrix} -Ae_1 & -Ae_2 & \cdots & -Ae_n, & Ae_1 & \cdots, & Ae_n \\ e_1 & e_2, & \cdots & e_n, & e_1, & \cdots, & e_n \end{pmatrix}$

$= \begin{pmatrix} -A & A \\ I & I \end{pmatrix}$

$\Rightarrow \quad P^{-1} M P = \begin{pmatrix} 0 & 0 \\ 0 & 2I \end{pmatrix}.$ #.

六、【8分】假设A和B都是n阶正定矩阵. 证明:

$$\det A \cdot \det B \leq \left(\frac{1}{n}\mathrm{Tr}\,(AB)\right)^n$$

证明:

(1) A 正定，存在正定 C 使得

$$A = C^2$$

(2) $AB = C^2B$.

$$\det AB = \det C^2B = \det CBC$$

$$\mathrm{Tr}\,AB = \mathrm{Tr}\,C^2B = \mathrm{Tr}\,CBC.$$

且 CBC 正定.

(3) 书上习题

$$\det A \leq \left(\frac{1}{n}\mathrm{Tr}\,A\right)^n$$

$$\Rightarrow \quad \det CBC \leq \left(\frac{1}{n}\mathrm{Tr}\,CBC\right)^n$$

$$\Rightarrow \quad \det AB \leq \left(\frac{1}{n}\mathrm{Tr}\,AB\right)^n \quad \#.$$