

期末考试 Ver.

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- 混合积: $[\vec{u} \vec{v} \vec{w}] = (\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$ \Rightarrow 平行六面体的体积
- 记: $\begin{cases} g_{ij} = \vec{g}_i \cdot \vec{g}_j \\ g^{ij} = \vec{g}^i \cdot \vec{g}^j \end{cases}$ 对偶积: $\vec{g}^i \cdot \vec{g}_j = \delta_j^i \Rightarrow \vec{g}^i = \frac{1}{\sqrt{g}} (\vec{g}_j \times \vec{g}_k)$ and $\sqrt{g} = [\vec{g}_1 \vec{g}_2 \vec{g}_3]$
 - 指标升降: $\vec{u} = u^i \vec{g}_i = u_i \vec{g}^i$; $u_i = g_{ij} u^j$ and $u^i = g^{ij} u_j$
- 曲线坐标系
 - $\vec{r} = \vec{r}(x^1, x^2, x^3) \Rightarrow d\vec{r} = \vec{g}_i dx^i$ where $\vec{g}_i = \frac{\partial \vec{r}}{\partial x^i}$ 为曲线坐标系自然基矢量, 其对偶基一般不存在。
 - $\nabla x^i \cdot \vec{g}_i = \frac{\partial x^i}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta_j^i \Rightarrow \vec{g}^i = \nabla x^i \Rightarrow \vec{g}_i = \nabla x_i$ (if exists)
- 坐标转换
 - 基矢量: $\vec{g}_{i'} = \beta_{i'}^i \vec{g}_i$, $\vec{g}^{i'} = \beta_i^{i'} \vec{g}^i$, $\beta_{i'}^i \beta_i^{i'} = \delta_j^i$, $\beta_{i'}^i = \frac{\partial x^i}{\partial x^{i'}}$
 - 分量转换: $u_{i'} = \beta_{i'}^i u_i$, $u^{i'} = \beta_i^{i'} u^i$
 - 矢量的定义: $\vec{u} = u_i \vec{g}^i = u^i \vec{g}_i = u_{i'} \vec{g}^{i'} = u^{i'} \vec{g}_{i'}$
 - 并矢
 - $\vec{u}\vec{v} = u^i v^j \vec{g}_i \vec{g}_j = u_i v_j \vec{g}^i \vec{g}^j$;
- 并矢式: 几个并矢的线性组合
 - 缩并: 两个基矢量点积缩并: $\vec{g}^i \cdot \vec{g}^k = g^{ik}$
- 张量
 - 度量矩阵: $g_{ij} \vec{g}^i \vec{g}^j = g_{i'j'} \vec{g}^{i'} \vec{g}^{j'} = G$
 - 置换张量: $\epsilon_{ijk} = [\vec{g}_i \vec{g}_j \vec{g}_k] = \sqrt{g} \epsilon_{ijk}$; $\epsilon^{ijk} = \frac{1}{\sqrt{g}} \epsilon^{ijk}$;
 ϵ_{ijk} 不是张量!
 - Kronecker 张量: $\delta = \epsilon \epsilon \Rightarrow \delta^{ijk} = \begin{vmatrix} \delta_r^i & \delta_s^i & \delta_t^i \\ \delta_r^j & \delta_s^j & \delta_t^j \\ \delta_r^k & \delta_s^k & \delta_t^k \end{vmatrix} = \epsilon^{ijk} \epsilon_{rst} = \epsilon_{ijk} \epsilon^{rst}$
- 常用关系
 - $\epsilon^{ijk} \epsilon_{ist} = \delta_{ist}^{ijk} = \delta_s^j \delta_t^k - \delta_s^k \delta_t^j$
 - $\epsilon^{ijk} \epsilon_{ijt} = 2\delta_t^k$; $\epsilon^{ijk} \epsilon_{ijk} = 2\delta_k^k = 6$
- 矢积:
 - $\vec{g}_i \times \vec{g}_j = \epsilon_{ijl} \vec{g}^l = \vec{g}_i \vec{g}_j \cdot \vec{\epsilon}$
 - $\vec{a} \times \vec{b} = \vec{a} \vec{b} \cdot \vec{\epsilon} = \vec{\epsilon} \cdot \vec{a} \vec{b}$
- 混合积
 - $[\vec{a} \vec{b} \vec{c}] = a^i b^j c^k \epsilon_{ijk}$
 - $\left| T^i_j \right| = \frac{1}{6} \delta_{lmn}^{ijk} T^l_i T^m_j T^n_k$
- 连续介质应力张量
 - 面元空间到面力空间的线性变换, $\sigma: \{\delta A\} \rightarrow \{\delta F\}$
 - σ 对称且 $\vec{f} = \vec{\sigma} \cdot \vec{n}$ 为单位面积上的表面力

out

$$(\nabla \cdot \vec{\sigma}) + \rho \vec{f}_{out} = \rho \frac{d\vec{v}}{dt}$$

○ 应变张量: $d\vec{r} \rightarrow d\hat{\vec{r}}: d\hat{\vec{r}} \cdot d\hat{\vec{r}} - d\vec{r} \cdot d\vec{r} = 2\varepsilon_{ij} dx^i dx^j$

- 拉格朗日坐标不变, 变的是基

$$\varepsilon_{ij} = \frac{1}{2}(\hat{g}_{ij} - g_{ij})$$

• 二阶张量

○ 不变量:

$$\phi_1 = T^i_i = \vec{G} \cdot \vec{T} = \text{tr}(\vec{T})$$

$$\phi_2 = \frac{1}{2}(T^i_i T^l_l - T^i_l T^l_i) = \frac{1}{2}[(\vec{G} \cdot \vec{T})^2 - \vec{T} \cdot \vec{T}]$$

$$\phi_3 = \det \vec{T} = \det [T^i_j] = \det \begin{bmatrix} T^1_1 & T^1_2 & T^1_3 \\ T^2_1 & T^2_2 & T^2_3 \\ T^3_1 & T^3_2 & T^3_3 \end{bmatrix} = \frac{1}{6} \varepsilon^{ijk} \varepsilon_{lmn} T^l_i T^m_j T^n_k$$

$$\text{▪ 矩: } \phi_n^* = \text{tr}(\vec{T}^n)$$

$$\square \phi_1^* = \phi_1; \phi_2^* = (\phi_1)^2 - 2\phi_2; \phi_3^* = (\phi_1)^3 - 3\phi_1\phi_2 + 3\phi_3$$

$$\square \phi_1 = \phi_1^*; \phi_2 = \frac{1}{2}[(\phi_1^*)^2 - \phi_2^*]; \phi_3 = \frac{1}{6}(\phi_1^*)^3 - \frac{1}{2}\phi_1^*\phi_2^* + \frac{1}{3}\phi_3^*$$

○ 线性变换是一个二阶张量

$$\circ [\vec{T} \cdot \vec{u} \vec{T} \cdot \vec{v} \vec{T} \cdot \vec{w}] = (\det \vec{T})[\vec{u} \vec{v} \vec{w}]$$

○ $\det \vec{T} \neq 0 \rightarrow$ 正则 否则为退化

○ 标准型

- 三不等实根

$$\vec{T} = \lambda_1 \vec{g}_1 \vec{g}^1 + \lambda_2 \vec{g}_2 \vec{g}^2 + \lambda_3 \vec{g}_3 \vec{g}^3$$

- 一实根二复根

$$\vec{T} = \lambda_1 \vec{g}_1 \vec{g}^1 + \lambda_2 \vec{g}_2 \vec{g}^2 + \lambda_3 \vec{g}_3 \vec{g}^3$$

$$\vec{g}'_1 = \vec{g}_1 + \vec{g}_2; \vec{g}'_2 = i(\vec{g}_1 - \vec{g}_2); \vec{g}'_3 = \vec{g}_3$$

$$\vec{T} = (\lambda \vec{g}'_1 + \mu \vec{g}'_2) \vec{g}'^1 + (-\mu \vec{g}'_1 + \lambda \vec{g}'_2) \vec{g}'^2 + \lambda_3 \vec{g}'_3 \vec{g}'^3$$

- 二重实根

若当标准型

$$\vec{T} = \lambda_1 \vec{g}_1 \vec{g}^1 + (\lambda_1 \vec{g}_2 + \vec{g}_1) \vec{g}^2 + \lambda_3 \vec{g}_3 \vec{g}^3$$

- 三重根

$$\vec{T} = \lambda_1 \vec{g}_1 \vec{g}^1 + (\lambda_1 \vec{g}_2 + \vec{g}_1) \vec{g}^2 + (\lambda_1 \vec{g}_3 + \vec{g}_2) \vec{g}^3$$

○ 特殊二阶张量

- 反对称张量 $\Omega^\dagger = -\Omega$

$$\text{实标准型: } \vec{\Omega} = -\phi \vec{e}_1 \vec{e}_2 + \phi \vec{e}_2 \vec{e}_1$$

$$\text{反偶矢量: } \vec{\omega} = -\frac{1}{2} \vec{\varepsilon} : \vec{\Omega} = \phi \vec{e}_3 \Rightarrow \vec{\Omega} = -\vec{\varepsilon} \cdot \vec{\omega}$$

- 正交张量: $Q^{-1} = Q^T$

$$\vec{Q} = \vec{e}'_1 \vec{e}_1 + \vec{e}'_2 \vec{e}_2 + \vec{e}'_3 \vec{e}_3$$

- 各向同性张量: 在任意笛卡尔直角坐标系中, 其分量不随坐标的正交转换而改变的张量

$$\square \text{二阶各向同性张量: } \lambda G \text{ 即球形张量}$$

$$\square \text{三阶各向: } \lambda \varepsilon \text{ (仅对非反射坐标正交变换)}$$

$$\square \text{四阶: } \lambda_1 g_{ij} g_{kl} \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l + \lambda_2 g_{ik} g_{jl} \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l + \lambda_3 g_{il} g_{jk} \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l$$

• 张量函数

○ 应不依赖于基或坐标系的选取, 则此函数也叫各向同性张量函数。

$$\chi = f(\vec{X}_1, \dots, \vec{X}_n) \rightarrow \tilde{\chi} = f(\tilde{\vec{X}}_1, \dots, \tilde{\vec{X}}_n)$$

$$X = X^{ijk\dots} \vec{g}_i \vec{g}_j \vec{g}_k \dots \rightarrow \tilde{X} = X^{ijk\dots} (\vec{Q} \cdot \vec{g}_i) (\vec{Q} \cdot \vec{g}_j) \dots$$

- 矢量的标量函数

$\phi = f(|\vec{v}|)$ (充要条件)

- 二阶适量的标量函数

$\phi = f(\vec{T})$ ϕ 是 \vec{T} 的标量不变量函数

$$T = \frac{1}{2}(T + T^T) + \frac{1}{2}(T - T^T) = N + \Omega$$

$$\phi = f(trN, tr(N^2), tr(N^3), tr(\Omega^2), tr(N \cdot \Omega^2), tr(N^2 \cdot \Omega^2), tr(N^2 \cdot \Omega^2 \cdot N \cdot \Omega))$$

这七个量并不独立，但无法选出六个独立的标量不变量把其他的标量表示成这六个的函数。

- 二阶张量的二阶张量函数

$$\tilde{H} = \phi(T) = a_0 G + a_1 T + \dots + a_n T^n$$

$\{a_i\}$ 是 T 的变量函数

- Hamilton-Cayley 等式

$$\Delta(T) = T^3 - \phi_1^T T^2 + \phi_2^T T - \phi_3^T G = 0$$

ϕ_i^T 为 T 主不变量

- 对收敛的幂级数 $\phi(T)$ 有

$$\tilde{H} = a'_0 G + a'_1 T + a'_2 T^2$$

- 对称张量函数 $\tilde{H} = f(N)$ 为各向同性的充要条件为

$$\tilde{H} = f(N) = k_0 G + k_1 N + k_2 N^2$$

$$k_i = k_i(\phi_1^N, \phi_2^N, \phi_3^N)$$

- 张量函数导数

- 对于增量 \vec{u} 的有限微分

$$\vec{F}'(\vec{v}; \vec{u}) = \lim_{h \rightarrow 0} \frac{1}{h} [\vec{F}(\vec{v} + h\vec{u}) - \vec{F}(\vec{v})]$$

其对 \vec{u} 为线性关系

- 定义 $\vec{F}'(\vec{v}; \vec{u}) = \vec{F}'(\vec{v}) \cdot \vec{u}$

此处的点积为顺序点(由后往前末点末)

则 $\vec{F}'(\vec{v})$ 称为 $\vec{F}(\vec{v})$ 的导数，写作 $\frac{d\vec{F}(\vec{v})}{d\vec{v}}$

- 导函数的阶数

若 \vec{F} 是 m 阶张量， \vec{v} 是 n 阶张量，则导函数是 $m+n$ 阶张量

- 导函数计算公式

$$\vec{F}'(\vec{v}) = \frac{\partial F_{ij\dots}}{\partial v_{mn\dots}} \vec{g}^i \vec{g}^j \dots \vec{g}_m \vec{g}_n \dots = \frac{\partial F_{ij\dots}}{\partial v_{n\dots}^m} \vec{g}^i \vec{g}^j \dots \vec{g}^m \vec{g}_n \dots$$

- 链式法则

对复合函数 $\tilde{H}^{(p)}(\vec{T}^{(m)}) = \vec{G}^{(p)}(\vec{F}^{(n)}(\vec{T}^{(m)}))$

则 $\tilde{H}'^{(p+m)}(\vec{T}^{(m)}) = \vec{G}'^{(p+n)}(\vec{F}^{(n)}) \cdot \vec{F}'^{(n+m)}(\vec{T}^{(m)})$

□ 矢量的标量函数: $f'(\vec{v}) = \frac{\partial f}{\partial v_{i'}} \vec{g}_{i'}$

□ 矢量的矢量函数: $\vec{F} = \frac{\partial F_i}{\partial v_j} \vec{g}^i \vec{g}_j \rightarrow d\vec{F} = F'(\vec{v}) \cdot d\vec{v}$

□ 矢量的二阶张量函数: $\vec{T}'(\vec{v}) = \frac{\partial T_{ij}^k}{\partial v_k} \vec{g}_i \vec{g}^j \vec{g}^k = \frac{\partial T_{ij}}{\partial v_k} \vec{g}^i \vec{g}^j \vec{g}_k$

□ 梯度散度旋度

◆ $\nabla \equiv \frac{\partial}{\partial v^i} \vec{g}^i$

◆ 梯度: $\vec{F}\nabla = \frac{\partial \vec{F}}{\partial v^i} \vec{g}^i = \vec{F}'(\vec{v}), \nabla \vec{F} = \vec{g}^i \frac{\partial \vec{F}}{\partial v^i}$

- ◆ 散度: $\nabla \cdot \vec{T} = \vec{g}^i \cdot \frac{\partial \vec{T}}{\partial v^i}$, $\vec{T} \cdot \nabla = \frac{\partial \vec{T}}{\partial v^i} \cdot g^i$
- ◆ 旋度: $\nabla \times \vec{T} = \vec{g}^i \times \frac{\partial \vec{T}}{\partial v^i} = \vec{\varepsilon} \cdot (\nabla \vec{T})$
- $\vec{T} \times \nabla = \frac{\partial \vec{T}}{\partial v^i} \times \vec{g}^i = (\vec{T} \nabla) \cdot \vec{\varepsilon}$

□ 二阶张量的标量函数导数

- ◆ $f'(\vec{T}) = \frac{\partial f}{\partial T_{ij}} \vec{g}_i \vec{g}_j$, T各分量独立
- ◆ 对称张量: $f'(\vec{S}) = \frac{df[\frac{1}{2}(S+S^T)]}{d\vec{S}}$

□ 二阶张量不变量的导数

- ◆ $\frac{d\phi_k^*}{d\vec{T}} = k(\vec{T}^{k-1})^T$
- ◆ $\frac{d\phi_1}{d\vec{T}} = \vec{G}$
- ◆ $\frac{d\phi_2}{d\vec{T}} = \phi_1 \vec{G} - \vec{T}^T$
- ◆ $\frac{d\phi_3}{d\vec{T}} = \phi_3(T^{-1})^T = (\phi_2 \vec{G} - \phi_1 \vec{T} + \vec{T}^2)^T$

□ 二阶张量函数的导数

- ◆ $\vec{H}'(\vec{T}) = \frac{\partial H^{ij}}{\partial T^{kl}} \vec{g}_i \vec{g}_j \vec{g}^k \vec{g}^l \Rightarrow d\vec{H} = \vec{H}'(\vec{T}) \cdot d\vec{T}$

○ 基矢量的导数

- 当基矢量也随着自变量变化时, 求导将变得复杂
- Christoffel 符号

- 第一类: $\frac{\partial \vec{g}_j}{\partial x^i} = \Gamma_{ij,k} \vec{g}^k$
- 第二类: $\frac{\partial \vec{g}_j}{\partial x^i} = \Gamma_{ij}^k \vec{g}_k$, $\frac{\partial \vec{g}^j}{\partial x^i} = -\Gamma_{ik}^j \vec{g}^k$

▪ Christoffel 符号的性质:

- $\Gamma_{ij}^k = \Gamma_{ji}^k$
- $\Gamma_{ij}^k \vec{g}^i \vec{g}^j \vec{g}_k$ 不是张量
- $\Gamma_{ij}^k = \Gamma_{ij,l} g^{kl}$
- $\Gamma_{ij,k} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$
- $\frac{\partial \vec{g}_j}{\partial x^i} = \Gamma_{ij}^k \vec{g}_k \Rightarrow \frac{\partial \vec{g}^j}{\partial x^i} = -\Gamma_{ik}^j \vec{g}^k$
- $\Gamma_{i'j'}^{k'} = \frac{\partial^2 x^p}{\partial x^{i'} \partial x^{j'}} \frac{\partial x^{k'}}{\partial x^p} + \beta_{i'}^p \beta_{j'}^q \beta_{k'}^r \Gamma_{pq}^r$

▪ 张量分量对坐标的协变导数

- $\vec{T}'(\vec{r}) = \vec{T} \nabla = T_{..k;l}^{ij} \vec{g}_i \vec{g}_j \vec{g}^k \vec{g}^l$, $T_{..k;l}^{ij} = \frac{\partial T^{ij}}{\partial x^l} + T_{..k}^{mj} \Gamma_{ml}^i + T_{..k}^{im} \Gamma_{ml}^j - T_{..m}^{ij} \Gamma_{kl}^m$
- $\nabla \vec{T} = \vec{g}^l \frac{\partial \vec{T}}{\partial x^l} = \nabla_l T_{..k}^{ij} \vec{g}^l \vec{g}_i \vec{g}_j \vec{g}^k$

□ 协变导数的性质

- ◆ 度量分量 G 的任何分量的协变导数为零

$$\nabla_i g_{jk} = 0; \nabla_i \delta_k^j = 0, \nabla_i g^{jk} = 0$$

- ◆ 置换张量 ε 的分量的协变导数为零

$$\nabla_l \varepsilon^{ijk} = 0; \nabla_l \varepsilon_{ijk} = 0$$

- ◆ 对张量分量进行缩并与求协变导数次数可以调换

- ◆ 两个张量分量乘积的协变导数满足

$ij \quad ij \quad ij$

$$\nabla_s(A^{ij}{}_k B^l{}_m) = (\nabla_s A^{ij}{}_k) B^l{}_m + A^{ij}{}_k (\nabla_s B^l{}_m)$$

- 张量场函数的散度与梯度 $\nabla = \vec{g}^s \frac{\partial}{\partial x^s}$

□ 梯度: $\vec{T}\nabla, \nabla\vec{T}$

□ 散度: $\vec{T} \cdot \nabla = \frac{\partial \vec{T}}{\partial x^s} \cdot \vec{g}^s; \nabla \cdot \vec{T} = \vec{g}^s \cdot \frac{\partial \vec{T}}{\partial x^s}$

□ 旋度: $\nabla \times \vec{T} = \vec{\epsilon} : (\nabla \vec{T}); \vec{T} \times \nabla = (\vec{T} \nabla) : \vec{\epsilon}$

□ 矢量场

◆ 散度: $\text{div } \vec{F} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} F^m)}{\partial x^m}$

◆ 旋度: $\text{curl } \vec{F} = \epsilon^{ijk} \partial_i F_j g_k$

◆ Laplace: $\nabla^2 \vec{F} = g^{sr} F_{ijk;sr} \vec{g}^i \vec{g}^j \vec{g}^k$

- 积分定理

□ $\oint_a d\vec{a} = 0$

□ $\int_V d\nu \nabla \vec{F} = \oint_a d\vec{a} \cdot \vec{F}; \int_V d\nu \nabla \cdot \vec{F} = \oint_a d\vec{a} \cdot \vec{F}; \int_V d\nu \nabla \times \vec{F} = \oint_a d\vec{a} \times \vec{F}$

□ $\int_a d\vec{a} \cdot (\nabla \times \vec{F}) = \oint_l d\vec{l} \cdot \vec{F}$

连续介质基本方程: $\nabla \cdot \vec{\sigma} + \rho \vec{f} = \rho \vec{w} = \frac{\rho D \vec{v}}{Dt}$

○ 曲面知识

- 自然基矢量

□ $\vec{\rho}_\alpha = \frac{\partial \vec{r}}{\partial \xi^\alpha}, \alpha = 1, 2$

□ $d\vec{\rho} = \vec{\rho}_\alpha d\xi^\alpha$

□ 完备基: $\left\{ \vec{\rho}_1, \vec{\rho}_2, \vec{n} = \frac{\vec{\rho}_1 \times \vec{\rho}_2}{|\vec{\rho}_1 \times \vec{\rho}_2|} \right\} \Rightarrow \{ \vec{\rho}^1, \vec{\rho}^2, \vec{n} \}$

- Riemann-Christoffel 张量(曲率张量)

□ Euclidean空间: 平直空间, 包含三维可展曲面, 即空间可以找到一组坐标使得 $g_{ik} \equiv const$, 即 $\Gamma_{ij}^k \equiv 0$

□ Riemann空间: 弯曲空间, 指找不到一组坐标使得上述描述成立

□ Riemann-Christoffel张量

$$R_{rsq}^p = \frac{\partial \Gamma_{rq}^p}{\partial x^s} - \frac{\partial \Gamma_{rs}^p}{\partial x^q} + \Gamma_{rq}^t \Gamma_{ts}^p - \Gamma_{rs}^t \Gamma_{tq}^p; \Gamma_{ij}^k = \frac{1}{2} g^{ks} \left(\frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s} \right)$$

若 $R_{rsq}^p \neq 0$, 则该点处无法展开为 Euclidean 空间

$$R_{ijkl} = \frac{\partial \Gamma_{jl,i}}{\partial x^k} - \frac{\partial \Gamma_{jk,i}}{\partial x^l} + \Gamma_{jk}^r \Gamma_{il,r} - \Gamma_{jl}^r \Gamma_{ik,r}$$

性质

$$\bullet R_{ijkl} = -R_{jikl}; R_{ijkl} = -R_{ijlk}; R_{ijkl} = R_{klji}$$

- 曲面的第一基本张量

□ $a_{\alpha\beta} = \vec{\rho}_\alpha \cdot \vec{\rho}_\beta, [a^{\alpha\beta}] = [a_{\alpha\beta}]^{-1}$

□ 第一基本型: $I = (ds)^2 = d\vec{\rho} \cdot d\vec{\rho} = a_{\alpha\beta} d\xi^\alpha d\xi^\beta$

□ 第一基本张量: $\vec{a} = a_{\alpha\beta} \vec{\rho}^\alpha \vec{\rho}^\beta = \delta_\alpha^\beta \vec{\rho}^\alpha \vec{\rho}^\beta$

□ 对切平面内矢量: $S = S_\alpha \rho^\alpha; \vec{a} \cdot \vec{S} = \vec{S} \cdot \vec{a} = \vec{S}$

- 曲面的第二基本张量 (反应弯曲程度)

□ $b_{\alpha\beta} = \vec{n} \cdot \frac{\partial^2 \vec{\rho}}{\partial \xi^\alpha \partial \xi^\beta} = \vec{n} \cdot \frac{\partial \vec{\rho}_\alpha}{\partial \xi^\beta} = -\frac{\partial \vec{n}}{\partial \xi^\beta} \cdot \rho_\alpha$

□ $\vec{b} = b_{\alpha\beta} \vec{\rho}^\alpha \vec{\rho}^\beta$

- 第二基本型，切平面上领域点到曲面的距离

$$II = b_{\alpha\beta} d\xi^\alpha d\xi^\beta$$

▪ 曲面的主曲率

- 法截面：通过曲面的法线 \vec{n} 所作的截面（无穷多个）

- 法截面曲线：法截面与曲面相交的曲线（无穷多个）

□ 曲率： $\kappa = -\frac{II}{I} = \frac{2\delta}{(ds)^2} = -\frac{b_{\alpha\beta} d\xi^\alpha d\xi^\beta}{a_{\lambda\gamma} d\xi^\lambda d\xi^\gamma}$, κ 的值取决于 $d\xi^1/d\xi^2$ 。

其比值变化时，能得到 κ 的极值，这就是主曲率。

法截面曲线的切向单位矢量： $\vec{t} = \frac{d\vec{\rho}}{ds} = \vec{\rho}_\alpha \frac{d\xi^\alpha}{ds} = t^\alpha \vec{\rho}_\alpha$

And thus $\kappa = -\vec{t} \cdot \vec{b} \cdot \vec{t}$, 用拉格朗日乘子法($\vec{t} \cdot \vec{t} = 1$)，有 $(\vec{b} - \lambda \vec{a}) \cdot \vec{t} = 0$

两个特征方向 $t_{(1)}, t_{(2)}$ 为两个特征方向（主方向）。

而法截面曲率 κ 的两个极值为 $\begin{cases} \kappa_1 = -\lambda_{(1)} \\ \kappa_2 = -\lambda_{(2)} \end{cases}$

- 平均曲率： $H = \kappa_1 + \kappa_2 = -\phi_1^b = -b_\alpha^\alpha$

Gauss曲率： $K = \kappa_1 \kappa_2 = \lambda_{(1)} \lambda_{(2)} = \det(\vec{b})$

▪ 旋转张量

- $c_{\alpha\beta} = [\vec{\rho}_\alpha \vec{\rho}_\beta \vec{n}], c_{11} = c_{22} = 0, c_{12} = -c_{21} = \sqrt{a},$
 $a = [\vec{\rho}_\alpha \vec{\rho}_\beta \vec{n}]^2 = \det(a_{\alpha\beta})$

□ 特殊关系

- ◆ $\vec{\rho}_\alpha \times \vec{\rho}_\beta = c_{\alpha\beta} \vec{n}$
- ◆ $\vec{\rho}^\alpha \times \vec{\rho}^\beta = c^{\alpha\beta} \vec{n}$
- ◆ $\vec{\rho}^\alpha \times \vec{\rho}_\beta = c_\beta^\alpha \vec{n}$
- ◆ $\vec{\rho}_\alpha \times \vec{\rho}^\beta = c_\alpha^\beta \vec{n}$

□ 表示矢积

- ◆ $\vec{v} = v^\alpha \vec{\rho}_\alpha; \vec{w} = w^\beta \vec{\rho}_\beta \Rightarrow \vec{v} \times \vec{w} = (\vec{v} \cdot \vec{c} \cdot \vec{w}) \vec{n} = (\vec{v} \vec{w} : \vec{c}) \vec{n}$

□ $c - \delta$ 等式

- ◆ $c_{\alpha\beta} c^{\gamma\gamma} = (\vec{\rho}_\alpha \times \vec{\rho}_\beta) \cdot (\vec{\rho}^\gamma \times \vec{\rho}^\gamma) = \delta_\alpha^\gamma \delta_\beta^\gamma - \delta_\alpha^\gamma \delta_\beta^\gamma$

- ◆ $(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$

▪ 曲面基本矢量的求导公式

- 法向矢量对坐标的导数

Weingarten 公式： $\frac{\partial \vec{n}}{\partial \xi^\alpha} = -b_{\alpha\beta}^\beta \vec{\rho}_\beta$

- 基矢量对坐标的导数(Christoffel 符号)

$$\frac{\partial \vec{\rho}_\alpha}{\partial \xi^\beta} = \frac{\partial \vec{\rho}_\beta}{\partial \xi^\alpha} = \dot{\Gamma}_{\alpha\beta}^\gamma \vec{\rho}_\gamma + b_{\alpha\beta} \vec{n}$$

$$\frac{\partial \vec{\rho}^\gamma}{\partial \xi^\beta} = -\dot{\Gamma}_{\alpha\beta}^\gamma \vec{\rho}^\alpha + b_{\beta\gamma}^\gamma \vec{n}$$

$$\dot{\Gamma}_{\alpha\beta}^\gamma = \frac{1}{2} a^{\gamma\lambda} \left(\frac{\partial a_{\alpha\lambda}}{\partial \xi^\beta} + \frac{\partial a_{\beta\lambda}}{\partial \xi^\alpha} - \frac{\partial a_{\alpha\beta}}{\partial \xi^\lambda} \right)$$

$$\dot{\Gamma}_{\alpha\beta\lambda}^\gamma = \dot{\Gamma}_{\alpha\beta}^\gamma a_{\gamma\lambda} = \frac{1}{2} \left(\frac{\partial a_{\alpha\lambda}}{\partial \xi^\beta} + \frac{\partial a_{\beta\lambda}}{\partial \xi^\alpha} - \frac{\partial a_{\alpha\beta}}{\partial \xi^\lambda} \right)$$

□ 第一张量逆变分量的导数

$$\frac{\partial a^{\alpha\beta}}{\partial \xi^\lambda} = -a^{\alpha\mu} \dot{\Gamma}_{\lambda\mu}^\beta - a^{\beta\mu} \dot{\Gamma}_{\lambda\mu}^\alpha$$

$$\frac{\partial \sqrt{a}}{\partial \xi^\alpha} = \dot{\Gamma}_{\alpha\beta}^\beta \sqrt{a}$$

▪ 曲面基本方程

$$\alpha\beta \quad \alpha\gamma \quad \alpha\beta \quad \alpha\gamma$$

- Codazzi 方程: $b_{\alpha\beta;\gamma} = b_{\alpha\gamma;\beta}$ or $\dot{\nabla}_\gamma b_{\alpha\beta} = \dot{\nabla}_\beta b_{\alpha\gamma}$
- Gauss 方程: $\dot{R}_{\alpha\gamma\beta}^\lambda = b_{\alpha\beta}b_{\gamma}^\lambda - b_{\alpha\gamma}b_{\beta}^\lambda$ or $\dot{R}_{\alpha\beta\gamma\nu} = b_{\alpha\gamma}b_{\beta\nu} - b_{\alpha\nu}b_{\beta\gamma}$
Where $\dot{R}_{\alpha\gamma\beta}^\lambda = \frac{\partial \dot{r}_{\alpha\beta}^\lambda}{\partial \xi^\gamma} - \frac{\partial \dot{r}_{\alpha\gamma}^\lambda}{\partial \xi^\beta} + \dot{\Gamma}_{\alpha\beta}^\mu \dot{\Gamma}_{\mu\gamma}^\lambda - \dot{\Gamma}_{\alpha\gamma}^\mu \dot{\Gamma}_{\mu\beta}^\lambda$

▪ 曲面上场函数的导数

- $\dot{\nabla} = \vec{\rho}^\alpha \frac{\partial}{\partial \xi^\alpha}$
- $\dot{\nabla} f = \vec{\rho}^\alpha \frac{\partial f}{\partial \xi^\alpha}, df = d\vec{\rho} \cdot \dot{\nabla} f$
- $\dot{\nabla} \vec{v} = \vec{\rho}^\alpha \frac{\partial \vec{v}}{\partial \xi^\alpha}, d\vec{v} = d\vec{\rho} \cdot \dot{\nabla} \vec{v}$
While $\frac{\partial \vec{v}}{\partial \xi^\lambda} = (v_{;\lambda}^\alpha - v^3 b_\lambda^\alpha) \vec{\rho}_\alpha + \left(\frac{\partial v^3}{\partial \xi^\lambda} + v^\omega b_{\omega\lambda} \right) \vec{n}$
 $\dot{\nabla} \vec{v} = (v_{;\lambda}^\alpha - v^3 b_\lambda^\alpha) \vec{\rho}^\lambda \vec{\rho}_\alpha + \left(\frac{\partial v^3}{\partial \xi^\lambda} + v^\omega b_{\omega\lambda} \right) \vec{\rho}^\lambda \vec{n}$
 $(v_{;\lambda}^\alpha - v^3 b_\lambda^\alpha) \vec{\rho}^\lambda \vec{\rho}_\alpha = (v_{\alpha;\lambda} - v^3 b_{\lambda\alpha}) \vec{\rho}^\lambda \vec{\rho}^\alpha$ 为切面矢量场
- $\dot{\nabla} \cdot \vec{v} = v_{;\alpha}^\alpha - v^3 b_\alpha^\alpha$
- $\dot{\nabla} \times \vec{v} = c^{\alpha\beta} v_{\beta;\alpha} \vec{n} - c^{\alpha\beta} \left(\frac{\partial v^3}{\partial \xi^\alpha} + v_\omega b_\alpha^\omega \right) \vec{\rho}_\beta$
- $\dot{\nabla} \dot{T} = \rho^\alpha \frac{\partial \dot{T}}{\partial \xi^\alpha}$, e.g. $\dot{\nabla} a = b_{\alpha\beta} (\vec{\rho}^\alpha \vec{n} \vec{\rho}^\beta + \vec{\rho}^\alpha \vec{\rho}^\beta \vec{n}), \dot{\nabla} c = b_\lambda^\omega c_{\alpha\omega} (\vec{\rho}^\lambda \vec{\rho}^\alpha \vec{n} - \vec{\rho}^\lambda \vec{n} \vec{\rho}^\alpha)$

▪ 等距曲面 (平行曲面)

- 若曲面沿法向距离参考曲面的距离为 z , 则有该曲面上的矢径可表示为
 $\vec{r} = \vec{\rho}(\xi^1, \xi^2) + z \vec{n}(\xi^1, \xi^2)$
- 基矢量: $\vec{r}_\alpha = \frac{\partial \vec{r}}{\partial \xi^\alpha} = (\delta_\alpha^\omega - z b_\alpha^\omega) \vec{\rho}_\omega = f_\alpha^\omega \rho_\omega$
 $f_\alpha^\omega = \delta_\alpha^\omega - z b_\alpha^\omega$ 为两个曲面基矢量的变换系数

□ 第一基本型

$$\tilde{I} = (d\tilde{s})^2 = g_{\tilde{\alpha}\tilde{\beta}} d\xi^\alpha d\xi^\beta, \tilde{g} = g_{\tilde{\alpha}\tilde{\beta}} \tilde{r}^{\tilde{\alpha}} \tilde{r}^{\tilde{\beta}} = \tilde{a}$$

□ 第三基本型

$$g_{\tilde{\alpha}\tilde{\beta}} = a_{\alpha\beta} - 2zb_{\alpha\beta} + z^2 v_{\alpha\beta}$$

$$v_{\alpha\beta} = b_\alpha^\omega b_\beta^\lambda a_{\omega\lambda} \Rightarrow \tilde{b} \cdot \tilde{b} = \tilde{v} = v_{\alpha\beta} \vec{\rho}^\alpha \vec{\rho}^\beta$$

$$III = v_{\alpha\beta} d\xi^\alpha d\xi^\beta$$

$$\tilde{b} \cdot \tilde{b} + H\tilde{b} + K\tilde{a} = 0$$

• 质点运动

- 对质点上的任意矢量: $\vec{u}(t) = u^i(t) g_i(x^k(t))$
- $\frac{d\vec{u}(t)}{dt} = \left(\frac{du^i}{dt} + u^m v^k \Gamma_{mk}^i \right) \vec{g}_i = \frac{Du^i}{Dt} \vec{g}_i$
- $v^k = \frac{dx^k}{dt} \Rightarrow \vec{v} = v^k \vec{g}_k$
 $\frac{Du_i}{Dt} = \left(\frac{du_i}{dt} - u_m v^k \Gamma_{ik}^m \right) \Rightarrow \frac{Du_i}{Dt} = g_{ij} \frac{Du^j}{Dt}; \frac{Du^i}{Dt} = g^{ij} \frac{Du_j}{Dt}$
注意: $\frac{du_i}{dt} \neq g_{ij} \frac{du^j}{dt}; \frac{du^i}{dt} \neq g^{ij} \frac{du_j}{dt}$

○ 坐标

▪ Lagrange 坐标

其为嵌在物体质点上的, 随着物体一起运动和变形的目标, 又叫随体目标, 记作 ξ^i , 自然基矢量可能时有关

▪ Euler 坐标

x^i 固定在空间中的参考坐标, 又称空间坐标, 自然基矢量时无关

▪ 坐标变换: $(t, x^i) \leftrightarrow (t^\xi, \xi^i)$

$$\begin{cases} t = t^\xi \\ x^i = x^i(t^\xi, \xi^i) \end{cases} \Leftrightarrow \begin{cases} t^\xi = t \\ \xi^i = \xi^i(t, x^i) \end{cases}$$

$$\text{速度场: } \frac{\partial \vec{r}}{\partial t^\xi} = \frac{\partial \vec{r}}{\partial t} \frac{\partial t}{\partial t^\xi} + \frac{\partial \vec{r}}{\partial x^i} \frac{\partial x^i}{\partial t^\xi} = \frac{\partial x^i}{\partial t^\xi} \hat{g}_i = \frac{d \vec{r}}{dt} = \frac{dx^i}{dt} \hat{g}_i$$

$$\text{注意: } \frac{\partial \vec{r}}{\partial t} = \frac{\partial \vec{r}}{\partial t}|_{x^i} = 0 \Rightarrow \frac{\partial}{\partial t^\xi} = \left(\frac{\partial}{\partial t} \right)_{\xi^i} = \frac{d}{dt}$$

▪ 拉格朗日坐标的自然基矢量

$$\hat{g}_i = \frac{\partial x^j}{\partial \xi^i} \hat{g}_j; \hat{g}_i = \frac{\partial \xi^j}{\partial x^i} \hat{g}_j; \frac{\partial \hat{g}_i}{\partial t^\xi} \neq \frac{\partial \hat{g}_i}{\partial t} \neq 0$$

$$\hat{v}^i = \frac{\partial \xi^i}{\partial x^j} v^j; v^i = \frac{\partial x^i}{\partial \xi^j} \hat{v}^j; \hat{v}_i = \hat{g}_{ij} \hat{v}^j \text{ 其中 } \hat{g}_{ij} = \hat{g}_i \cdot \hat{g}_j$$

▪ 基矢量的物质导数 (注意: $\hat{v} = \nabla$)

$$\frac{\partial \hat{g}_i}{\partial t^\xi} = \frac{\partial \hat{v}}{\partial \xi^i} = (\hat{v} \hat{v}) \hat{g}_k = (\hat{v} \hat{v}) \cdot \hat{g}_i \Rightarrow \frac{\partial \hat{g}^i}{\partial t^\xi} = -(\hat{v} \hat{v}) \cdot \hat{g}^i$$

□ 度量张量

$$\frac{\partial \hat{g}_{ij}}{\partial t^\xi} = \hat{v}_j \hat{v}_i + \hat{v}_i \hat{v}_j$$

□ 应变率张量

$$\vec{d} = \hat{d}_{ij} \hat{g}^i \hat{g}^j = \frac{1}{2} (\hat{v}_j \hat{v}_i + \hat{v}_i \hat{v}_j) \hat{g}^i \hat{g}^j = \frac{1}{2} (\hat{v} \hat{v} + \hat{v} \hat{v}) = \frac{1}{2} (\vec{v} \nabla + \nabla \vec{v})$$

□ 速度分解

$$\vec{v} \nabla = \frac{1}{2} (\vec{v} \nabla + \nabla \vec{v}) + \frac{1}{2} (\vec{v} \nabla - \nabla \vec{v}) = \vec{d} + \vec{\nabla} \Rightarrow \nabla \vec{v} = \vec{d} - \vec{\Omega}$$

$$d \vec{v} = (\vec{v} \nabla) \cdot d \vec{r} = \vec{d} \cdot d \vec{r} + \vec{\omega} \times d \vec{r} \Rightarrow \vec{\omega} = -\frac{1}{2} \vec{\varepsilon} : \vec{\Omega}$$

□ Euler基矢量: $\frac{\partial \hat{g}_i}{\partial t^\xi} = v^j \Gamma_{ij}^k \hat{g}_k, \frac{\partial \hat{g}^i}{\partial t^\xi} = -v^j \Gamma_{jk}^i \hat{g}^k$

▪ 矢量场的导数

$$\square \frac{d \vec{u}}{dt} = \frac{\partial \hat{u}^i}{\partial t^\xi} \hat{g}_i + \hat{u}_m \hat{d}_{.m}^i \hat{g}_i + \hat{u}_m \hat{\Omega}_{.m}^i \hat{g}_i = \frac{\partial \hat{u}_i}{\partial t^\xi} \hat{g}^i - \hat{u}^m \hat{d}_{im} \hat{g}^i + \hat{u}^m \hat{\Omega}_{im} \hat{g}^i$$

$$\square \frac{d \vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \vec{v} \cdot \nabla \vec{u} = \partial \vec{u} / \partial t^\xi$$

▪ 变形梯度张量 $\{d \vec{r}^0\} \rightarrow \{d \hat{r}\}$

□ 变形梯度张量: $d \vec{r} = \vec{F} \cdot d \vec{r}^0 \Rightarrow \hat{g}_i = \vec{F} \cdot \vec{g}_i^0 \Rightarrow \vec{F} = \hat{g}_i \hat{g}^{0i}, F^{-1} = \hat{g}_i^0 \hat{g}^i$

□ 物质体元变换公式: $d \hat{v} = \phi_3^F d v^0$

□ 物质有向面元: $d \hat{a} = \phi_e^F \vec{F}^{-T} \cdot d \hat{a}^0; (\vec{T} \cdot \vec{u}) \times (\vec{T} \cdot \vec{v}) = \phi_e^T \vec{T}^{-T} \cdot (\vec{u} \times \vec{v})$

▪ 物质导数

$$\square \frac{\partial \vec{F}}{\partial t^\xi} = (\vec{v} \hat{v}) \cdot \vec{F}$$

▪ Green应变张量

$$\square \vec{E} = \frac{1}{2} (\hat{g}_{ij} - g_{ij}^0) \hat{g}^{0i} \hat{g}^{0j}$$

$$\square \vec{F}^{-T} \cdot \frac{\partial \vec{E}}{\partial t^\xi} \cdot \vec{F}^{-1} = \vec{d}$$

题目集

2021年12月21日 20:42

1. 形如 $T_{ij}\vec{g}^i\vec{g}^j$ 的张量所构成的线性空间是几维的，其中 $i, j = 1, 2, 3$ 。

解：

$$\dim(V \otimes V) = \dim(V) \cdot \dim(V) = 3 \times 3 = 9 \text{ 维}$$

2. 简述四阶张量 C_{ijkl} 的 Voigt 对称性，并写出二维空间中 C_{ijkl} 的所有独立分量。

解：

Voigt 对称性即指 C_{ijkl} 满足如下对称：

$$C_{ijkl} = C_{jikl}$$

$$C_{ijkl} = C_{ijlk}$$

$$C_{ijkl} = C_{lkij}$$

可见前后两个指标必须成对交换， C_{ijkl} 共有 16 个分量

$$C_{1212} = C_{2112} = C_{1221} = C_{2121}$$

$$C_{1112} = C_{1121} = C_{2111} = C_{1211}$$

$$C_{2212} = C_{2221} = C_{1222} = C_{2122}$$

$$C_{1122} = C_{2211};$$

$$C_{1111}, C_{2222}$$

共 5 个独立分量。

3. 有一个二阶张量 \vec{A} ，其在笛卡尔坐标系下的分量 $[a_{ij}] = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ ，相应的基记作 $\vec{i}_1, \vec{i}_2, \vec{i}_3$ ，现给出一组新基 $\vec{g}_1 = \vec{i}_1 + \vec{i}_3, \vec{g}_2 =$

$\vec{i}_1 + \vec{i}_2 + \vec{i}_3, \vec{g}_3 = -\vec{i}_1 + \vec{i}_2 + \vec{i}_3$ 。试写出相应基矢量转换系数，并求 \vec{A} 在新基下的协变分量和逆变分量。

解：

$$\text{由 } \vec{g}_1 = \beta_{1'}^i \vec{i}_i = \vec{i}_1 + \vec{i}_3 \Rightarrow \begin{cases} \beta_{1'}^1 = 1 \\ \beta_{1'}^2 = 0 \\ \beta_{1'}^3 = 1 \end{cases} \text{ 同理 } \begin{cases} \beta_{2'}^1 = 1 \\ \beta_{2'}^2 = 1 \\ \beta_{2'}^3 = 1 \end{cases} \begin{cases} \beta_{3'}^1 = -1 \\ \beta_{3'}^2 = 1 \\ \beta_{3'}^3 = 1 \end{cases}$$

$$\text{可令: } [\beta_{j'}^i] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ (规定上指标为行)}$$

协变分量：

$$\begin{aligned} [a_{i'j'}] &= [\beta_{i'}^i \beta_{j'}^j a_{ij}] = [\beta_{i'}^i]^T [a_{ij}] [\beta_{j'}^j]^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 4 \\ 2 & 1 & 4 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 4 \\ 6 & 7 & 3 \\ -4 & -3 & 1 \end{bmatrix} \end{aligned}$$

逆变分量满足：

$$[a^{i'j'}] = [\beta_i^{i'} \beta_j^{j'} a^{ij}] = [\beta_i^{i'}][a^{ij}] [\beta_j^{j'}]^T$$

笛卡尔坐标下有 $[a^{ij}] = [a_{ij}]$

$$\text{又有: } \beta_i^{i'} \beta_i^j = \delta_i^j = [\beta_i^j][\beta_i^{i'}]$$

$$\Rightarrow [\beta_i^{i'}] = [\beta_i^j]^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$[\beta_i^{i'}] = [\beta_i^i][a^{ij}][\beta_j^{j'}]^T = \begin{bmatrix} 0 & -1 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -\frac{5}{2} & \frac{1}{2} \\ 3 & -2 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

4. 试证明 $\delta_j^i \vec{g}_i \vec{g}^j$ 是张量：

解：

$$jk \quad ik \quad jk$$

$$\delta_j^i \vec{g}_i \vec{g}^j = g_j^i \vec{g}_i \vec{g}^j = g_{jk} (g^{ik} \vec{g}_i) \vec{g}^j = g_{jk} \vec{g}^k \vec{g}^j = \vec{G}$$

故相当于证明 \vec{G} 是张量。

在一组基下, $\vec{G} = g_{ij} \vec{g}^i \vec{g}^j$, $g_{ij} = \vec{g}_i \cdot \vec{g}_j$

在另一组基下, $\vec{G} = g_{i'j'} \vec{g}^{i'} \vec{g}^{j'}$

$$\begin{aligned} g_{ij} \vec{g}^i \vec{g}^j &= g_{ij} (\beta_{i'}^i \vec{g}^{i'}) (\beta_{j'}^j \vec{g}^{j'}) = (\vec{g}_i \cdot \vec{g}_j) (\beta_{i'}^i \vec{g}^{i'}) (\beta_{j'}^j \vec{g}^{j'}) \\ &= (\beta_{i'}^i \vec{g}_i \cdot \beta_{j'}^j \vec{g}_j) \vec{g}^{i'} \vec{g}^{j'} = (\vec{g}_{i'} \cdot \vec{g}_{j'}) \vec{g}^{i'} \vec{g}^{j'} = g_{i'j'} \vec{g}^{i'} \vec{g}^{j'} \end{aligned}$$

故 \vec{G} 的确为张量。

5. 试证 $\varepsilon_{ijk} \vec{g}^i \vec{g}^j \vec{g}^k$ 是张量

解:

由定义:

$$\begin{aligned} \varepsilon_{ijk} \vec{g}^i \vec{g}^j \vec{g}^k &= [\vec{g}_i \vec{g}_j \vec{g}_k] \vec{g}^i \vec{g}^j \vec{g}^k = [\vec{g}_i \vec{g}_j \vec{g}_k] (\beta_{i'}^i \vec{g}^{i'}) (\beta_{j'}^j \vec{g}^{j'}) (\beta_{k'}^k \vec{g}^{k'}) \\ &= [(\beta_{i'}^i \vec{g}_i) (\beta_{j'}^j \vec{g}_j) (\beta_{k'}^k \vec{g}_k)] \vec{g}^{i'} \vec{g}^{j'} \vec{g}^{k'} = [\vec{g}_{i'} \vec{g}_{j'} \vec{g}_{k'}] \vec{g}^{i'} \vec{g}^{j'} \vec{g}^{k'} \end{aligned}$$

6. 试用 ε_{ijk} 证明: 对3维空间中的二阶张量 \vec{A} 和 \vec{B} , 有 $\det(\vec{A} \cdot \vec{B}) = \det(\vec{A}) \det(\vec{B})$

解:

$$\begin{aligned} \text{有 } \det(\vec{A} \cdot \vec{B}) &= \det(A_{\cdot j}^i B_{\cdot k}^j \vec{g}_i \vec{g}^k) = A_{\cdot m}^i B_{\cdot 1}^m A_{\cdot n}^j B_{\cdot 2}^n A_{\cdot l}^k B_{\cdot 3}^l e_{ijk} \\ &= A_{\cdot m}^i A_{\cdot n}^j A_{\cdot l}^k \frac{(\varepsilon_{ijk})}{\sqrt{g}} B_{\cdot 1}^m B_{\cdot 2}^n B_{\cdot 3}^l \\ &= \det(\vec{A}) \frac{1}{\sqrt{g}} \varepsilon_{mnk} B_{\cdot 1}^m B_{\cdot 2}^n B_{\cdot 3}^l = \det(\vec{A}) \det(\vec{B}) \end{aligned}$$

7. 对已存在的坐标 x^i 可定义相应的自然基矢量 $\vec{g}_i = \frac{\partial \vec{r}}{\partial x^i}$, 利用对偶关系可以得到空间每点上的逆变基矢量 \vec{g}^i , 举例说明, 一般情况下, 并不存在与 \vec{g}^i 相对应的坐标。

解:

设有如下坐标关系

$$\begin{cases} x^I = x \\ x^{II} = (y + x^2)^{\frac{1}{3}} \end{cases} \Rightarrow \begin{cases} x = x^I \\ y = (x^{II})^3 - (x^I)^2 \end{cases} \Rightarrow \begin{cases} \vec{g}_1 = \frac{\partial \vec{r}}{\partial x^I} = \vec{i} - 2x^I \vec{j} \\ \vec{g}_2 = \frac{\partial \vec{r}}{\partial x^{II}} = 3(x^{II})^2 \vec{j} \end{cases}$$

由此可以得到对偶基

$$\begin{cases} \vec{g}^1 = \vec{i} \\ \vec{g}^2 = \frac{2x^I}{3(x^{II})^2} \left(\vec{i} + \frac{1}{2x^I} \vec{j} \right) = \frac{2x}{3(y + x^2)^{\frac{2}{3}}} \left(\vec{i} + \frac{1}{2x} \vec{j} \right) \end{cases}$$

但又有

$$\begin{cases} \vec{g}^1 = \frac{\partial \vec{r}}{\partial x_I} = \frac{\partial x}{\partial x_I} \vec{i} + \frac{\partial y}{\partial x_I} \vec{j} \\ \vec{g}^2 = \frac{\partial \vec{r}}{\partial x_{II}} = \frac{\partial x}{\partial x_{II}} \vec{i} + \frac{\partial y}{\partial x_{II}} \vec{j} \end{cases}$$

由此可得雅可比式

$$\begin{aligned} \frac{\partial(x, y)}{\partial(x_I, x_{II})} &= \begin{bmatrix} 1 & \frac{2}{3}x(y + x^2)^{-\frac{2}{3}} \\ 0 & \frac{1}{3}(y + x^2)^{-\frac{2}{3}} \end{bmatrix} \\ &\Rightarrow \frac{\partial(x_I, x_{II})}{\partial(x, y)} = \begin{bmatrix} 1 & \frac{2}{3}x(y + x^2)^{-\frac{2}{3}} \\ 0 & \frac{1}{3}(y + x^2)^{-\frac{2}{3}} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2x \\ 0 & 3(y + x^2)^{\frac{2}{3}} \end{bmatrix} \end{aligned}$$

其有解的条件是, 二阶混合偏导与顺序无关, 但

$$\frac{\partial}{\partial y} \left(\frac{\partial x_I}{\partial x} \right) = 0 \neq \frac{\partial}{\partial x} \left(\frac{\partial x_I}{\partial y} \right) = -2$$

由此可见无解, 故不存在与 \vec{g}^i 对应的坐标。

8. 对已存在的坐标 x^i 可定义相应的自然基矢量 $\vec{g}_i = \frac{\partial \vec{r}}{\partial x^i}$, 利用对偶关系可以得到空间每点上的逆变基矢量 \vec{g}^i , 试证 $\vec{g}^i = \nabla x^i$

解:

$$\nabla x^i = \frac{\partial x^i}{\partial x} \vec{i} + \frac{\partial x^i}{\partial y} \vec{j} + \frac{\partial x^i}{\partial z} \vec{k}$$

$$\text{又 } \vec{g}_i = \frac{\partial x}{\partial x^i} \vec{i} + \frac{\partial y}{\partial x^i} \vec{j} + \frac{\partial z}{\partial x^i} \vec{k}$$

$$\text{故得到: } \vec{g}_i \cdot \nabla x^j = \left(\frac{\partial x^j}{\partial x^i} \right) = \delta_i^j$$

$$\text{根据定义: } \vec{g}^j \cdot \vec{g}_i = \delta_i^j \Rightarrow \vec{g}^j = \nabla x^j$$

9. 对给定的欧几里德空间中的坐标 x^i , 试问其对应的“协变坐标” x_i 在什么条件下存在? 如存在, 该怎么求?

解:

$$dx_i = g_{ij} dx^j = (\vec{g}_i \cdot \vec{g}_j) dx^j = \vec{g}_i \cdot d\vec{r} \Rightarrow \vec{g}_i = \frac{dx_i}{d\vec{r}} = \nabla x_i$$

故当且仅当三个自然基矢量场 $\{\vec{g}_i\}$ 是无旋的, 上述方程才有解。

如果该条件满足, 则协变坐标可通过路径积分求解:

$$x_i(A) = \int_0^A \vec{g}_i \cdot d\vec{r}$$

10. 为什么应力可以表示成一个对称二阶张量

解:

某点上单位面积上表面力 \vec{f} 是单位法向量 \vec{n} 的函数, $\vec{f} = \vec{f}(\vec{n})$

考虑如右图所示的微元体, 根据牛顿力学公式有

$$\vec{a} = \frac{\sum \delta \vec{F}_i}{\rho \delta V} + \vec{g} = \frac{\sum \vec{f}_i \delta A_i}{\rho \delta V} + \vec{g}$$

$$\text{即: } \vec{f}_3 = \frac{(\vec{a} - \vec{g}) \rho \delta V}{\delta A_3} - \frac{\delta A_1}{\Delta A_3} \vec{f}_1 - \frac{\delta A_2}{\Delta A_3} \vec{f}_2$$

$$\text{当 } \delta l \rightarrow 0 \text{ 时, 第一项为高阶无穷小, 故有: } \vec{f}_3 = -\frac{\delta A_1}{\delta A_3} \vec{f}_1 - \frac{\delta A_2}{\delta A_3} \vec{f}_2$$

$$\text{又由几何知识可得: } \sum \vec{n}_i \delta A_i = 0 \Rightarrow \vec{n}_3 = -\frac{\delta A_1}{\delta A_3} \vec{n}_1 - \frac{\delta A_2}{\delta A_3} \vec{n}_2$$

故可得:

$$f(\vec{n}_3) = f \left(-\frac{\delta A_1}{\delta A_3} \vec{n}_1 - \frac{\delta A_2}{\delta A_3} \vec{n}_2 \right) = \vec{f}_3 = -\frac{\delta A_1}{\delta A_3} \vec{f}(\vec{n}_1) - \frac{\delta A_2}{\delta A_3} \vec{f}(\vec{n}_2)$$

由此可见 $\vec{f} = \vec{f}(\vec{n})$ 是一个线性函数, 由线性代数知识可知:

可以找到一个 3×3 的矩阵 σ 使得若

$$\vec{f} = f^i \vec{e}_i \Rightarrow f^i = \sigma(i, j) n^j$$

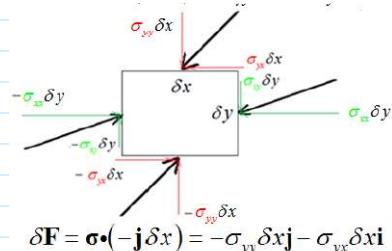
由于 n^j 为任意单位法向量分量, p^i 为矢量分量, 根据商法则, σ 一定是一个二阶张量, 可记作: $\vec{f} = \vec{\sigma} \cdot \vec{n}$ 。 $\vec{\sigma}$ 即为应力。

下证其为对称矩阵:

不失一般性, 可考虑二维情况

$$\text{在笛卡尔坐标系下, } \vec{\sigma} = \sigma_{xx} \vec{i} + \sigma_{yx} \vec{j} + \sigma_{xy} \vec{j} + \sigma_{yy} \vec{j}$$

考虑如图的微元体:



其合力矩: $\delta M = (\sigma_{yx} - \sigma_{xy}) \delta x \delta y \sim (\sigma_{yx} - \sigma_{xy})(\delta l)^n$

转动惯量: $\delta I = \int_{\delta V} r^2 \rho dV \sim \rho (\delta l)^2 (\delta l)^n$

由 $\delta M = \omega \delta I$ 可得: $\omega \sim \frac{(\sigma_{yx} - \sigma_{xy})}{\rho (\delta l)^2}$

当 $\delta l \rightarrow 0$ 时, 若 ω 有限, 则 $(\sigma_{yx} - \sigma_{xy}) \sim o((\delta l)^2) \rightarrow 0$

故有: $\sigma_{xy} = \sigma_{yx}$

综上所述, 应力可以表示为一个对称二阶张量 $\vec{\sigma}$

11. 在笛卡尔坐标下写出纯剪切情况下的应力矩阵, 已知 $|\vec{\sigma}_1| = |\vec{\sigma}_3| = \tau$, $|\vec{\sigma}_2| = 0$ 。其中 $\vec{\sigma}_i$ 表示面 e_i 上单位面积所受的力。

解:

$$\text{由于是纯剪切力, 故有 } \sigma = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 0 \end{bmatrix}$$

由于 $|\vec{\sigma}_2| = |\vec{\sigma} \cdot \vec{e}_2| = 0 \Rightarrow \sigma_{12} = \sigma_{23} = 0$

又 $|\vec{\sigma}_3| = |\vec{\sigma} \cdot \vec{e}_3| = \tau = |\vec{\sigma}_1| \Rightarrow \sigma_{13} = \pm \tau$

$$\therefore \sigma = \begin{bmatrix} 0 & 0 & \pm \tau \\ 0 & 0 & 0 \\ \pm \tau & 0 & 0 \end{bmatrix}$$

12. 有二阶张量 \vec{T}, \vec{S} , 对任意矢量 \vec{a} 有 $\vec{T} \cdot \vec{a} = \vec{a} \cdot \vec{S}$, 试证明 $\vec{T} = \vec{S}^T$.

解:

$$\begin{aligned} \vec{T} \cdot \vec{a} &= T_{ij} \vec{g}^i \vec{g}^j \cdot a^k \vec{g}_k = T_{ij} a^k \vec{g}^i \delta_k^j = T_{ik} a^k \vec{g}^i \\ \vec{a} \cdot \vec{S} &= a^k \vec{g}_k \cdot (S_{ij} \vec{g}^i \vec{g}^j) = a^k S_{kj} \vec{g}^j = S_{ki} a^k \vec{g}^i \end{aligned}$$

由于对任意 \vec{a} 都成立, 故一定有: $T_{ik} = S_{ki}$

故: $\vec{T} = \vec{S}^T$

13. 问: $\vec{n} \cdot \vec{G}(\vec{v} \cdot \vec{T}) = \vec{v} \cdot (\vec{T} \vec{n})^T$ 式中转置符号 T 的具体含义 (其中 \vec{G} 是度规张量, \vec{n}, \vec{v} 是一阶张量, \vec{T} 是二阶张量)。

解:

$$\begin{aligned} \vec{n} \cdot \vec{G}(\vec{v} \cdot \vec{T}) &= n_i \vec{g}^i \cdot \vec{g}_k \vec{g}^k (\vec{v} \cdot \vec{T}) \\ &= n_i \vec{g}^i (v_j \vec{g}^j \cdot T^{mn} \vec{g}_m \vec{g}_n) \\ &= n_i \vec{g}^i v_j T^{jn} \vec{g}_n \\ &= v_j T^{jn} n_i \vec{g}^i \vec{g}_n \\ &= v_j (\vec{g}^j \cdot \vec{g}_k) T^{kn} n^i \vec{g}_i \vec{g}_n \\ &= \vec{v} \cdot (T^{kn} n^i \vec{g}_k \vec{g}_i \vec{g}_n) \\ &= \vec{v} \cdot (T^{ki} n^n)^T \vec{g}_k \vec{g}_i \vec{g}_n \end{aligned}$$

故转置符号 T 表示转置 \vec{n} 后两个指标。

14. 试写出二阶张量的 Hamilton-Cayley 等式。

解:

二阶张量的 Hamilton-Cayley 等式为

$$\Delta(\vec{T}) = \vec{T}^3 - \phi_1^T \vec{T}^2 + \phi_2^T \vec{T} - \phi_3^T \vec{G} = 0$$

其中: $\phi_1 = \text{tr}(\vec{T})$; $\phi_2 = \frac{1}{2}(T_i^l T_l^i - T_l^i T_i^l)$; $\phi_3 = \det \vec{T}$

证明见书本 p101-102

无重根时, 注意到二阶张量 \vec{T} 的特征多项式:

$$\Delta(\lambda) = \lambda^3 - \phi_1^T \lambda^2 + \phi_2^T \lambda - \phi_3^T = 0$$

则可以选择一组基对角化 \vec{T}

则此时

$$\begin{aligned} \Delta(\vec{T}) &= \text{diag}\{\lambda_i^3\} - \phi_1^T \text{diag}\{\lambda_i^2\} + \phi_2^T \text{diag}\{\lambda_i\} - \phi_3^T \text{diag}\{1\} \\ &= \text{diag}\{\lambda_i^3 - \phi_1^T \lambda_i^2 + \phi_2^T \lambda_i - \phi_3^T\} = \text{diag}\{0 \dots\} = 0 \end{aligned}$$

15. 试证明: 实对称二阶张量 \vec{N} 的特征值均为实数, 且对应不同特征值的特征方向相互正交。

解:

对特征向量有

$$\vec{N} \cdot \vec{a} = \lambda \vec{a}$$

取共轭得到: $\vec{N} \cdot \vec{a} = \bar{\lambda} \vec{a}$

两边点乘 \vec{a} , 且因 $\vec{a} \cdot \vec{N} = \vec{N} \cdot \vec{a} = \lambda \vec{a}$ 有

$\lambda \vec{a} \cdot \vec{a} = \bar{\lambda} \vec{a} \cdot \vec{a} \Rightarrow \lambda = \bar{\lambda}$, 故特征值为实数

又对两个不同的特征向量有

$$\begin{cases} \vec{N} \cdot \vec{a}_1 = \lambda_1 \vec{a}_1 \Rightarrow \vec{a}_2 \cdot \vec{N} \cdot \vec{a}_1 = \lambda_1 \vec{a}_1 \cdot \vec{a}_2 \\ \vec{N} \cdot \vec{a}_2 = \lambda_2 \vec{a}_2 \Rightarrow \vec{a}_1 \cdot \vec{N} \cdot \vec{a}_2 = \lambda_2 \vec{a}_2 \cdot \vec{a}_1 \end{cases}$$

而由对称性可得: $\vec{a}_1 \cdot \vec{N} \cdot \vec{a}_2 = \vec{a}_2 \cdot \vec{N} \cdot \vec{a}_1$

故有: $\lambda_1 \vec{a}_1 \cdot \vec{a}_2 = \lambda_2 \vec{a}_2 \cdot \vec{a}_1 \Rightarrow \vec{a}_1 \cdot \vec{a}_2 = 0$

故不同特征值对应的特征矢量正交。

16. 将张量 $\vec{N} = \vec{e}_1 \vec{e}_1 + 2\vec{e}_2 \vec{e}_2 - 2(\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_1) - 2(\vec{e}_1 \vec{e}_3 + \vec{e}_3 \vec{e}_1)$ 化为标准型。

解：

$$\vec{N} \cdot \vec{a} = \lambda \vec{a}, \begin{bmatrix} 1 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\Delta(\lambda) = (\lambda - 4)(\lambda + 2)(\lambda - 1) \Rightarrow \lambda_1 = 4, \lambda_2 = -2, \lambda_3 = 1$$

$$\begin{bmatrix} -3 & -2 & -2 \\ -2 & -2 & 0 \\ -2 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \Rightarrow \text{取归一化 } \vec{a}_1 = -\frac{2}{3}\vec{e}_1 + \frac{2}{3}\vec{e}_2 + \frac{1}{3}\vec{e}_3$$

$$\begin{bmatrix} 3 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \Rightarrow \vec{a}_2 = \frac{2}{3}\vec{e}_1 + \frac{1}{3}\vec{e}_2 + \frac{2}{3}\vec{e}_3$$

$$\begin{bmatrix} 0 & -2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \Rightarrow \vec{a}_3 = -\frac{1}{3}\vec{e}_1 - \frac{2}{3}\vec{e}_2 + \frac{2}{3}\vec{e}_3$$

$$\vec{N} = \lambda_1 \vec{a}_1 \vec{a}_2 + \lambda_2 \vec{a}_2 \vec{a}_3 + \lambda_3 \vec{a}_3 \vec{a}_1$$

17. 什么叫“各向同性张量”？是证明：所有的各向同性二阶张量都可以写成 $p\vec{G}$

解：

欧几里得空间中的各向同性张量是一种在任意笛卡尔直角坐标系中，其分量不随坐标的正交转换而变换的张量。

任取一组单位正交基下，应用二阶张量的矩阵形式，其等效于

$$QAQ^{-1} = A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{取 } Q = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \Rightarrow a_{12} = a_{21} = 0$$

$$\text{取 } Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow a_{11} = a_{22}$$

故在这组基下，张量A的矩阵形式可以写为 $A = pI = pG$

$$\text{即: } \vec{A} = p\vec{G}$$

由于此关系与坐标选取无关，由张量定义可知上式在任何坐标系下均成立。

18. 写出度规张量 \vec{G} 和置换张量 $\vec{\epsilon}$ 的旋转量。

解：

对度规张量：

$$\tilde{G} = \vec{Q} \cdot \vec{G} \cdot \vec{Q}^T = \vec{Q} \cdot \vec{Q}^T = \vec{G}$$

对置换张量：

$$\begin{aligned} \tilde{\epsilon} &= (q_i^n q_j^m q_k^l \epsilon^{ijk}) \vec{g}_n \vec{g}_m \vec{g}_l \\ &= (q_i^n q_j^m q_k^l \epsilon^{ijk}) \vec{g}_n \vec{g}_m \vec{g}_l = (q_i^n q_j^m q_k^l \epsilon^{ijk}) \sqrt{g} \vec{g}_n \vec{g}_m \vec{g}_l \\ &= \det(\vec{Q}) \epsilon^{nml} \sqrt{g} \vec{g}_n \vec{g}_m \vec{g}_l = \det(\vec{Q}) \epsilon^{nml} \vec{g}_n \vec{g}_m \vec{g}_l = \det(\vec{Q}) \vec{\epsilon} \end{aligned}$$

19. 对任意阶张量 $T_{ijk\dots} \vec{g}^i \vec{g}^j \vec{g}^k \dots$ ，其旋转量是什么？

解：

按照定义，基矢量不变，分量主动变为正交变换后的分量：

$$\tilde{T} = q_i^n q_j^m q_k^l \dots T_{nml} \vec{g}^i \vec{g}^j \vec{g}^k \dots$$

其中 $\vec{Q} = q_i^j \vec{g}^i \vec{g}_j$ 为任意正交张量。

20. 若定义导函数 $F'(\vec{v})$ 由下式确定： $F'(\vec{v}; \vec{u}) = \vec{u} \cdot F'(\vec{v})$ ，试给出相应的求导公式：

解：

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{1}{h} [\vec{F}(\vec{v} + h\vec{u}) - \vec{F}(\vec{v})] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} [F_{ij\dots}(v_{mn\dots} \vec{g}^m \vec{g}^n \dots + h u_{mn\dots} \vec{g}^m \vec{g}^n \dots) \vec{g}^i \vec{g}^j \dots - F_{ij\dots}(v_{mn\dots} \vec{g}^m \vec{g}^n \dots) \vec{g}^i \vec{g}^j \dots] \\ &= \frac{\partial F_{ij\dots}}{\partial v_{mn\dots}} u_{mn\dots} \vec{g}^i \vec{g}^j \dots = u_{st\dots} (\vec{g}^s \vec{g}^t \dots) \cdot \left(\frac{\partial F_{ij\dots}}{\partial v_{mn\dots}} \vec{g}_m \vec{g}_n \dots \vec{g}^i \vec{g}^j \dots \right) = \vec{u} \cdot F'(\vec{v}) \\ &\text{所以: } \vec{F}'(\vec{v}) = \frac{\partial F_{ij\dots}}{\partial v_{mn\dots}} \vec{g}_m \vec{g}_n \dots \vec{g}^i \vec{g}^j \dots \end{aligned}$$

21. 对二阶张量 \vec{T} ，推导 $\frac{d\phi_k^*}{d\vec{T}}$ 。

解：

$$\frac{d\phi_k^*}{d\vec{T}} = \frac{d[\text{tr}(\vec{T}^k)]}{d\vec{T}}$$

不妨从定义出发推导

$$\begin{aligned} \phi_k^*(\vec{T} + h\vec{C}) &= \text{tr}(\vec{T} + h\vec{C})^k = \text{tr}[\vec{T}^k + h(\vec{C} \cdot \vec{T}^{k-1} + \vec{T} \cdot \vec{C} \cdot \vec{T}^{k-2} + \dots + \vec{T}^{k-1} \cdot \vec{C}) + o(h^2)] \\ &= \text{tr}(\vec{T}^k) + hk \text{tr}(\vec{T}^{k-1} \cdot \vec{C}) + o(h^2) \end{aligned}$$

$$\text{而 } \phi_k^*(\vec{T}; \vec{C}) = \lim_{h \rightarrow 0} \frac{1}{h} [\phi_k^*(\vec{T} + h\vec{C}) - \phi_k^*(\vec{T})] = \left(\frac{d\phi_k^*}{d\vec{T}} \right) : \vec{C}$$

$$\text{又 } \phi_k^*(\vec{T}; \vec{C}) = k \operatorname{tr}(\vec{T}^{k-1} \cdot \vec{C}) = k (\vec{T}^{k-1})^i_{\cdot j} C_i^j = k (\vec{T}^{k-1})^T : \vec{C}$$

$$\text{由此可得: } \left(\frac{d\phi_k^*}{d\vec{T}} \right) = k (\vec{T}^{k-1})^T$$

22. \vec{S} 是对称的二阶张量, 求 $\frac{d(\vec{S} \cdot \vec{S})}{d\vec{S}}$ 和 $\frac{d(\vec{S}\vec{S})}{d\vec{S}}$ 。

解:

$$\vec{S} \cdot \vec{S} = S_{ik} S^{kj} \vec{g}^i \vec{g}_j = g^{kl} S_{ik} S_{lj} \vec{g}^i \vec{g}^j$$

$$\frac{d(\vec{f}(\vec{S}))}{d\vec{S}} = \frac{d\left(\vec{f}\left(\frac{1}{2}(\vec{S} + \vec{S}^T)\right)\right)}{d\vec{S}}$$

$$\frac{d(\vec{S} \cdot \vec{S})}{d\vec{S}} = \frac{1}{4} g^{kl} \frac{\partial(S_{ik} S_{lj} + S_{il} S_{jl} + S_{ki} S_{lj} + S_{ki} S_{jl})}{\partial S_{mn}} \vec{g}^i \vec{g}^j \vec{g}_m \vec{g}_n$$

$$= \frac{1}{4} g^{kl} ((\delta_i^m \delta_k^n + \delta_k^m \delta_i^n)(S_{lj} + S_{jl}) + (S_{ki} + S_{ik})(\delta_l^m \delta_j^n + \delta_j^m \delta_l^n)) \vec{g}^i \vec{g}^j \vec{g}_m \vec{g}_n$$

$$= \frac{1}{4} g^{kl} (S_{lj} + S_{jl}) (\vec{g}^i \vec{g}^j \vec{g}_i \vec{g}_k + \vec{g}^i \vec{g}^j \vec{g}_k \vec{g}_i)$$

$$+ \frac{1}{4} g^{kl} (S_{ki} + S_{ik}) (\vec{g}^i \vec{g}^j \vec{g}_l \vec{g}_j + \vec{g}^i \vec{g}^j \vec{g}_j \vec{g}_l)$$

$$= \frac{1}{4} [(S_i^k + S_i^k) (\vec{g}^j \vec{g}^i \vec{g}_j \vec{g}_k + \vec{g}^j \vec{g}^i \vec{g}_k \vec{g}_j + \vec{g}^i \vec{g}^j \vec{g}_k \vec{g}_j + \vec{g}^i \vec{g}^j \vec{g}_j \vec{g}_k)]$$

$$\vec{S}\vec{S} = S_{ij} S_{kl} \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l$$

$$\frac{d(\vec{S}\vec{S})}{d\vec{S}} = \frac{1}{4} \frac{\partial(S_{ij} S_{kl} + S_{ji} S_{kl} + S_{ij} S_{lk} + S_{ji} S_{lk})}{\partial S_{mn}} \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l \vec{g}_m \vec{g}_n$$

$$= \frac{1}{4} ((\delta_i^m \delta_j^n + \delta_j^m \delta_i^n)(S_{kl} + S_{lk}) + (\delta_k^m \delta_l^n + \delta_l^m \delta_k^n)(S_{ij} + S_{ji})) \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l \vec{g}_m \vec{g}_n$$

$$= \frac{1}{4} (S_{kl} + S_{lk}) (\vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l \vec{g}_i \vec{g}_j + \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l \vec{g}_j \vec{g}_i)$$

$$+ \frac{1}{4} (S_{ij} + S_{ji}) (\vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l \vec{g}_k \vec{g}_l + \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l \vec{g}_l \vec{g}_k)$$

$$= \frac{1}{4} (S_{ij} + S_{ji}) (\vec{g}^k \vec{g}^l \vec{g}^i \vec{g}^j \vec{g}_k \vec{g}_l + \vec{g}^k \vec{g}^l \vec{g}^i \vec{g}^j \vec{g}_l \vec{g}_k + \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l \vec{g}_k \vec{g}_l + \vec{g}^i \vec{g}^j \vec{g}^k \vec{g}^l \vec{g}_l \vec{g}_k)$$

23. 试对自变量为对称二阶张量的张量函数 $f(\vec{S})$ 求导, 并说明所得导函数关于最后两个指标对称。

解:

(以下 \vec{S} 即 \vec{S})

\vec{S} 对称, 因此其只有一部分分量是独立的, 即 $\frac{\partial f}{\partial S^{ij}}$ 在数学分析中会失去意义。

按定义:

$$d\vec{f}(S^{ij}) = \frac{\partial \vec{f}}{\partial S^{ij}} \vec{g}^i \vec{g}^j : dS^{mn} \vec{g}_m \vec{g}_n = \vec{f}'(\vec{S}) : d\vec{S}$$

但由于任意一个反对称张量都有 $\vec{\Omega}: d\vec{S} = 0$:

故上式无法独立确定 $\vec{f}'(\vec{S})$, 而因此我们需要规定 $\vec{f}'(\vec{S}) = \frac{\partial \vec{f}}{\partial S^{ij}} \vec{g}^i \vec{g}^j$ 关于最后两个指标 i, j 对称。

或我们可以利用, 令 $\vec{X} = \frac{1}{2}(\vec{S} + \vec{S}^T)$, 对 $\vec{f}'(\vec{X})$ 求导:

$$\begin{aligned} \frac{d\vec{f}(\vec{X}(\vec{S}))}{d\vec{S}} &= \frac{d\vec{f}(\vec{X})}{d\vec{X}} : \frac{d\vec{X}}{d\vec{S}} = \frac{d\vec{f}(\vec{X})}{d\vec{X}} : \frac{1}{2} \frac{\partial(S^{ij} + S^{ji})}{\partial S^{mn}} \vec{g}_i \vec{g}_j \vec{g}^m \vec{g}^n \\ &= \frac{d\vec{f}(\vec{X})}{d\vec{X}} : \vec{g}_i \vec{g}_j (\vec{g}^i \vec{g}^j + \vec{g}^j \vec{g}^i) \end{aligned}$$

故可见关于最后两个指标 ij 对称。

24. 克氏记号是张量的分量吗? 试推导第二类克氏记号的坐标变换公式:

$$\Gamma_{i'j'}^{l'} = \frac{\partial^2 x^p}{\partial x^{i'} \partial x^{j'}} \frac{\partial x^{l'}}{\partial x^p} + \beta_i^p \beta_j^q \beta_r^{l'} \Gamma_{pq}^r$$

解:

克氏记号不是张量的分量, 因为不遵从坐标变换规律。

$$\begin{aligned} \Gamma_{i'j'}^{l'} &= \frac{\partial \vec{g}_{i'}^{l'}}{\partial x^{i'}} \cdot \vec{g}^{l'} = \frac{\partial}{\partial x^{i'}} \left(\frac{\partial x^p}{\partial x^{i'}} \vec{g}_p \right) \cdot \left(\frac{\partial x^{l'}}{\partial x^r} \right) \vec{g}^r \\ &= \left[\frac{\partial^2 x^p}{\partial x^{i'} \partial x^{j'}} \vec{g}_p + \frac{\partial x^p}{\partial x^{i'}} \left(\frac{\partial \vec{g}_p}{\partial x^q} \frac{\partial x^q}{\partial x^{j'}} \right) \right] \cdot \frac{\partial x^{l'}}{\partial x^r} \vec{g}^r \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial^2 x^p}{\partial x^{i'} \partial x^{j'}} \delta_p^r \frac{\partial x^{i'}}{\partial x^r} + \frac{\partial x^p}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{j'}} \frac{\partial x^{i'}}{\partial x^r} \Gamma_{pq}^r \\
&= \frac{\partial^2 x^p}{\partial x^{i'} \partial x^{j'}} \delta_p^r + \beta_{i'}^p \beta_j^q \beta_r^{i'} \Gamma_{pq}^r
\end{aligned}$$

25. 写出平面极坐标的自然基矢量 (用笛卡尔坐标系的基矢量 \vec{i}, \vec{j} 表示) 以及相应的Christoffel 符号。

解:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \Rightarrow \begin{cases} \vec{g}_r = \frac{\partial \vec{r}}{\partial r} = \cos \theta \vec{i} + \sin \theta \vec{j} \\ \vec{g}_\theta = \frac{\partial \vec{r}}{\partial \theta} = -r \sin \theta \vec{i} + r \cos \theta \vec{j} \end{cases}$$

$$\begin{cases} \frac{\partial \vec{g}_r}{\partial r} = 0 \\ \frac{\partial \vec{g}_r}{\partial \theta} = \frac{\vec{g}_\theta}{r} \end{cases} \Rightarrow \Gamma_{rr}^i = 0; \begin{cases} \Gamma_{r\theta}^r = 0 \\ \Gamma_{r\theta}^\theta = \frac{1}{r} \end{cases}$$

$$\begin{cases} \frac{\partial \vec{g}_\theta}{\partial r} = \frac{\vec{g}_\theta}{r} \\ \frac{\partial \vec{g}_\theta}{\partial \theta} = -r \vec{g}_r \end{cases} \Rightarrow \begin{cases} \Gamma_{r\theta}^r = 0 \\ \Gamma_{r\theta}^\theta = \frac{1}{r}; \Gamma_{\theta\theta}^\theta = -r \end{cases}$$

26. 写出球坐标的自然基矢量 (用笛卡尔坐标系中的基矢量 \vec{i}, \vec{j} 表示) 以及相应的Christoffel 符号。

解:

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \Rightarrow \begin{cases} \vec{g}_r = \sin \theta \cos \phi \vec{i} + \sin \theta \sin \phi \vec{j} + \cos \theta \vec{k} \\ \vec{g}_\theta = r \cos \phi \cos \theta \vec{i} + r \sin \phi \cos \theta \vec{j} - r \sin \theta \vec{k} \\ \vec{g}_\phi = -r \sin \theta \sin \phi \vec{i} + r \sin \theta \cos \phi \vec{j} \end{cases}$$

$$\frac{\partial \vec{g}_j}{\partial x^i} = \Gamma_{ij}^k \vec{g}_k$$

$$\begin{cases} \frac{\partial \vec{g}_r}{\partial r} = 0 \\ \frac{\partial \vec{g}_r}{\partial \theta} = \frac{1}{r} \vec{g}_\theta \Rightarrow \Gamma_{rr}^i = 0; \\ \frac{\partial \vec{g}_r}{\partial \phi} = \frac{1}{r} \vec{g}_\phi \end{cases} \begin{cases} \Gamma_{r\theta}^r = 0 \\ \Gamma_{r\theta}^\theta = \frac{1}{r}, \Gamma_{r\phi}^\phi = 0 \\ \Gamma_{r\theta}^\phi = 0 \end{cases}$$

$$\begin{cases} \frac{\partial \vec{g}_\theta}{\partial r} = \frac{1}{r} \vec{g}_\theta \\ \frac{\partial \vec{g}_\theta}{\partial \theta} = -r \vec{g}_r \Rightarrow \Gamma_{\theta\theta}^r = 0, \\ \frac{\partial \vec{g}_\theta}{\partial \phi} = \text{ctg } \theta \vec{g}_\phi \end{cases} \begin{cases} \Gamma_{\theta\theta}^\theta = -r \\ \Gamma_{\theta\theta}^\phi = 0, \Gamma_{\theta\phi}^\phi = 0 \\ \Gamma_{\theta\phi}^\theta = 0 \end{cases}$$

$$\begin{cases} \frac{\partial \vec{g}_\phi}{\partial r} = \frac{1}{r} \vec{g}_\phi \\ \frac{\partial \vec{g}_\phi}{\partial \theta} = \text{ctg } \theta \vec{g}_\phi \\ \frac{\partial \vec{g}_\phi}{\partial \phi} = -r \sin^2 \theta \vec{g}_r - \sin \theta \cos \theta \vec{g}_\theta \end{cases} \Rightarrow \begin{cases} \Gamma_{r\phi}^r = 0 \\ \Gamma_{r\phi}^\theta = 0, \Gamma_{r\phi}^\phi = \frac{1}{r} \\ \Gamma_{\theta\phi}^\phi = \text{ctg } \theta \end{cases} \begin{cases} \Gamma_{\phi\phi}^r = -r \sin^2 \theta \\ \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta \\ \Gamma_{\phi\phi}^\phi = 0 \end{cases}$$

27. 有三维空间中单位矢量 \vec{t} 和任意阶张量 \vec{T} , 证明 $\vec{T} = \vec{t}(\vec{t} \cdot \vec{T}) - \vec{t} \times (\vec{t} \times \vec{T})$

解:

$$\begin{aligned}
\vec{t} \times (\vec{t} \times \vec{T}) &= \vec{t} \times (\epsilon_{ijk} t^i T^j \cdots \vec{g}^k \vec{g}_{...}) = t_m t^i T^j \cdots \epsilon^{mkl} \epsilon_{ijk} \vec{g}_l \vec{g}_{...} \\
&= t_m t^i T^j \cdots \epsilon^{klm} \epsilon_{kij} \vec{g}_l \vec{g}_{...} = t_m t^i T^j \cdots (\delta_i^l \delta_j^m - \delta_j^l \delta_i^m) \vec{g}_l \vec{g}_{...} \\
&= t_m t^l T^m \cdots \vec{g}_l \vec{g}_{...} - t_i t^i T^l \cdots \vec{g}_l \vec{g}_{...} \\
&= t^l \vec{g}_l (t_m (\vec{g}^m \cdot \vec{g}_k) T^k \cdots) - \vec{T} \\
&= \vec{t}(\vec{t} \cdot \vec{T}) - \vec{T}
\end{aligned}$$

故有:

$$\vec{t}(\vec{t} \cdot \vec{T}) - \vec{t} \times (\vec{t} \times \vec{T}) = \vec{T}$$

28. 试证明: $\nabla^2(\vec{B} \cdot \vec{v}) = \nabla^2 \vec{B} \cdot \vec{v} + \vec{B} \cdot \nabla^2 \vec{v} + 2\nabla \vec{B} \cdot \nabla \vec{v}$, 并说明双点积的具体次序 (\vec{B} 是二阶张量, \vec{v} 是矢量)。

解:

这是一个标量, 则在 Euclidean 空间中与坐标选取无关, 故可选用直角坐标系, 则:

$$ij$$

$$\begin{aligned}
\nabla^2(\vec{B} \cdot \vec{v}) &= \nabla^2(B^{ij}v_j \vec{g}_i) = \nabla \cdot \left(\vec{g}^s \vec{g}_i \frac{\partial(B^{ij}v_j)}{\partial x^s} \right) \\
&= \nabla \cdot \left(v_j \vec{g}^s \vec{g}_i \frac{\partial B^{ij}}{\partial x^s} + B^{ij} \vec{g}^s \vec{g}_i \frac{\partial v_j}{\partial x^s} \right) \\
&= \vec{g}^r \cdot \frac{\partial}{\partial x^r} \left(v_j \vec{g}^s \vec{g}_i \frac{\partial B^{ij}}{\partial x^s} + B^{ij} \vec{g}^s \vec{g}_i \frac{\partial v_j}{\partial x^s} \right) \\
&= g^{rs} \vec{g}_i \frac{\partial v_j}{\partial x^r} \frac{\partial B^{ij}}{\partial x^s} + g^{rs} v_j \vec{g}_i \frac{\partial^2 B^{ij}}{\partial x^r \partial x^s} + g^{rs} \vec{g}_i \frac{\partial B^{ij}}{\partial x^r} \frac{\partial v_j}{\partial x^s} + g^{rs} B^{ij} \vec{g}_i \frac{\partial^2 v_j}{\partial x^r \partial x^s} \\
&= 2g^{rs} \vec{g}_i \frac{\partial v_j}{\partial x^r} \frac{\partial B^{ij}}{\partial x^s} + g^{rs} \left(\frac{\partial^2 B^{ij}}{\partial x^r \partial x^s} \right) \vec{g}_i (\vec{g}_j \cdot \vec{g}^k) v_k + g^{rs} B^{ij} \vec{g}_i (\vec{g}_j \cdot \vec{g}^k) \left(\frac{\partial^2 v_k}{\partial x^r \partial x^s} \right) \\
&= 2(\vec{g}^r \cdot \vec{g}^s) \left(\frac{\partial v_j}{\partial x^r} \right) (\vec{g}_j \cdot \vec{g}^k) \vec{g}_i \left(\frac{\partial B^{ik}}{\partial x^s} \right) + \nabla^2 \vec{B} \cdot \vec{v} + \vec{B} \cdot \nabla^2 \vec{v} \\
&= 2\vec{g}_i \left(\frac{\partial B^{ik}}{\partial x^s} \right) (\vec{g}^s \cdot \vec{g}^r) (\vec{g}_k \cdot \vec{g}^j) \left(\frac{\partial v_j}{\partial x^r} \right)
\end{aligned}$$

注意到：

$$\begin{aligned}
\nabla \vec{B} &= \left(\frac{\partial B^{ik}}{\partial x^s} \right) \vec{g}^s \vec{g}_i \vec{g}_k; \nabla \vec{v} = \left(\frac{\partial v_j}{\partial x^r} \right) \vec{g}^r \vec{g}^j \\
\text{故: } 2\vec{g}_i \left(\frac{\partial B^{ik}}{\partial x^s} \right) (\vec{g}^s \cdot \vec{g}^r) (\vec{g}_k \cdot \vec{g}^j) \left(\frac{\partial v_j}{\partial x^r} \right) &= 2\nabla \vec{B} \cdot \nabla \vec{v}
\end{aligned}$$

顺序为： $\vec{g}^s \vec{g}_i \vec{g}_k \dots \vec{g}^r \vec{g}^j = (\vec{g}^s \cdot \vec{g}^r) (\vec{g}_k \cdot \vec{g}^j) \vec{g}_i$

29. 补充完整以下张量公式，并证明。

$$\vec{u} \cdot (\nabla \cdot \vec{T}) = \nabla \cdot (\vec{T} \cdot \vec{u}) + \dots; \text{ 其中 } \vec{u} \text{ 是矢量, } \vec{T} \text{ 是二阶张量。}$$

解：

不妨取在直角坐标系中分析，因为协变导数与上述矢量和张量构成了一个矢量，其在坐标系变换下不变的

$$\begin{aligned}
\nabla \cdot (\vec{T} \cdot \vec{u}) &= \vec{g}^l \frac{\partial}{\partial x^l} \cdot (T_{ij} \vec{g}^i \vec{g}^j \cdot u_k \vec{g}^k) = \vec{g}^l \frac{\partial}{\partial x^l} \cdot (T^{ij} \vec{g}_i \delta_j^k u_k) \\
&= \vec{g}^l \frac{\partial}{\partial x^l} \cdot (\vec{g}_i T^{ik} u_k) = \frac{\partial}{\partial x^l} (T^{ik} u_k) \\
&= u_k \frac{\partial T^{ik}}{\partial x^l} + T^{ik} \frac{\partial u_k}{\partial x^l} \\
&= u_k (\vec{g}^k \cdot \vec{g}_j) \frac{\partial T^{jk}}{\partial x^l} + T^{ik} (\vec{g}_k \cdot \vec{g}^j) \frac{\partial u_j}{\partial x^l} \\
&= u_k (\vec{g}^k \cdot \vec{g}_j) (\vec{g}^l \cdot \vec{g}_l) \frac{\partial T^{lk}}{\partial x^l} + T^{lk} (\vec{g}_l \cdot \vec{g}^i) (\vec{g}_k \cdot \vec{g}^j) \frac{\partial u_j}{\partial x^l} \\
&= \vec{u} \cdot (\nabla \cdot \vec{T}) + \vec{T} : (\nabla \vec{u})
\end{aligned}$$

$$\vec{u} \cdot (\nabla \cdot \vec{T}) = \nabla \cdot (\vec{T} \cdot \vec{u}) + (-\vec{T} : (\nabla \vec{u})) \text{ [两边点乘]}$$

30. 对矩形区域证明Green变换公式： $\int_V dv \nabla \vec{T} = \oint_a d\vec{a} \vec{T}$ 。

解：

等式左边有：

$$\int_V dv \nabla \vec{T} = \int \sqrt{g} dx^1 dx^2 dx^3 \vec{g}^l \frac{\partial \vec{T}}{\partial x^l}$$

将该矩形区域分切为无穷小的矩形区域块，可以得到：

$$\int_{\Delta v} dv \nabla \vec{T} = \sqrt{g} dx^1 dx^2 dx^3 \vec{g}^l \frac{\partial \vec{T}}{\partial x^l}$$

对右侧在此小矩形区域中有：

$$\begin{aligned}
\oint_{\Delta a} da \vec{T} &= \left[\sqrt{g} \vec{g}^l \vec{T}_{(1)} + \frac{\partial}{\partial x^1} (\sqrt{g} \vec{g}^1) dx^1 \right] dx^2 dx^3 - \sqrt{g} \vec{g}^1 \vec{T}_{(1)} dx^2 dx^3 \\
&+ \left[\sqrt{g} \vec{g}^2 \vec{T}_{(2)} + \frac{\partial}{\partial x^2} (\sqrt{g} \vec{g}^2) dx^2 \right] dx^1 dx^3 - \sqrt{g} \vec{g}^2 \vec{T}_{(2)} dx^1 dx^3 \\
&+ \left[\sqrt{g} \vec{g}^3 \vec{T}_{(3)} + \frac{\partial}{\partial x^3} (\sqrt{g} \vec{g}^3) dx^3 \right] dx^1 dx^2 - \sqrt{g} \vec{g}^3 \vec{T}_{(3)} dx^1 dx^2 \\
&= \left[\frac{\partial (\sqrt{g} \vec{g}^l \vec{T})}{\partial x^l} dx^1 dx^2 dx^3 \right] = \left[\frac{\partial (\sqrt{g} \vec{g}^l)}{\partial x^l} \vec{T} + \sqrt{g} \vec{g}^l \frac{\partial \vec{T}}{\partial x^l} \right] dx^1 dx^2 dx^3
\end{aligned}$$

又因为 $\frac{\partial (\sqrt{g} \vec{g}^l)}{\partial x^l} = 0$

故在此微源矩形区域中有：

$$\int_{\Delta v} d\nu \nabla \vec{T} = \oint_{\Delta a} da \vec{T}$$

由于我们采用的是矩形分割，且本身积分区域为矩形区域，

显然有： $\sum \int_{\Delta v} d\nu \nabla \vec{T} = \int_V d\nu \nabla \vec{T}$; $\sum \oint_{\Delta a} da \vec{T} = \oint_a (da) \vec{T}$

则两边求和微元式即可得到：

$$\int_V d\nu \nabla \vec{T} = \oint_a (da) \vec{T}$$

31. 由二阶张量 \vec{T} 的散度 $\operatorname{div} \vec{T}$ 的物理定义出发，即由 $\operatorname{div} \vec{T} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S \vec{T} \cdot \vec{n} dS$, 导出 $\operatorname{div} \vec{T}$ 在直角笛卡尔坐标中的表达式

解：

$$\vec{T} = \vec{T}(x^i)$$

当 $V \rightarrow 0$ 时，体积近似于一个小立方块

$$\begin{aligned} \oint_S \vec{T} \cdot \vec{n} dS &\approx \vec{T}(x_0 + dx, y_0, z_0) \cdot \vec{x} dy dz - \vec{T}(x_0, y_0, z_0) \cdot \vec{x} dy dz \\ &+ \vec{T}(x_0, y_0, z_0 + dz) \cdot \vec{z} dx dy - \vec{T}(x_0, y_0, z_0) \cdot \vec{z} dx dy \\ &+ \vec{T}(x_0, y_0 + dy, z_0) \cdot \vec{y} dx dz - \vec{T}(x_0, y_0, z_0) \cdot \vec{y} dx dz \\ &= \left(\vec{T}(x_0, y_0, z_0) + \frac{\partial \vec{T}(x_0, y_0, z_0)}{\partial x} dx - \vec{T}(x_0, y_0, z_0) \right) \cdot \vec{x} dy dz \\ &\quad \left(\vec{T}(x_0, y_0, z_0) + \frac{\partial \vec{T}(x_0, y_0, z_0)}{\partial z} dz - \vec{T}(x_0, y_0, z_0) \right) \cdot \vec{z} dx dy \\ &\quad \left(\vec{T}(x_0, y_0, z_0) + \frac{\partial \vec{T}(x_0, y_0, z_0)}{\partial y} dy - \vec{T}(x_0, y_0, z_0) \right) \cdot \vec{y} dx dz \\ &= \frac{\partial \vec{T}}{\partial x^i} \cdot \vec{g}^i V \\ \operatorname{div} \vec{T} &= \frac{1}{V} \left(\frac{\partial \vec{T}}{\partial x^i} \cdot \vec{g}^i V \right) = \frac{\partial \vec{T}}{\partial x^i} \cdot \vec{g}^i \end{aligned}$$

32. 有一个关于参量 \vec{v} 的方程：

$$\left[\vec{v} + \sum_{i=1}^{\infty} \vec{M}_{i+1} \odot_i (\vec{v} \otimes_i \vec{v}) \right] \cdot \nabla \sum_{i=1}^{\infty} \vec{M}_{i+1} \odot_i (\vec{v} \otimes_i \vec{v}) = -\nabla \sum_{i=1}^{\infty} \vec{m}_i \odot_i (\vec{v} \otimes_i \vec{v}) + \nabla^2 \sum_{i=1}^{\infty} \vec{M}_{i+1} \odot_i (\vec{v} \otimes_i \vec{v})$$

其中 \vec{v} 是任一常矢量， \otimes_i 表示 i 重并， \odot_i 表示 i 重依次点积（ \odot_i 两边都从最后一个基本矢量开始，依次点积）， \vec{M}_{i+1} 为 $i+1$ 阶未知张量， \vec{m}_i 为 i 阶未知张量。

a. 考察等式两边一次 \vec{v} 的项，可得：

$$0 = -\nabla(\vec{m}_1 \odot_1 \vec{v}) + \nabla^2(\vec{M}_2 \odot_1 \vec{v}),$$

消去 \vec{v} 可得关于 \vec{m}_1 和 \vec{M}_2 的方程：

$$0 = -\nabla \vec{m}_1 + \nabla^2 \vec{M}_2$$

认为 \vec{m}_1 和 \vec{M}_2 可由此解出。

b. 考察等式两边二次 \vec{v} 的项，可得：

$$(\vec{v} + \vec{M}_2 \odot_1 \vec{v}) \cdot \nabla (\vec{M}_2 \odot_1 \vec{v}) = -\nabla(\vec{m}_2 \odot_2 \vec{v} \vec{v}) + \nabla^2(\vec{M}_3 \odot_2 \vec{v} \vec{v}), 消去 \vec{v} 可得到关于 \vec{m}_2 和 \vec{M}_3 的方程：$$

$\left[(\vec{I} + \vec{M}_2)^{T_1} \cdot \nabla \vec{M}_2 \right]^{T_2} = -\nabla \vec{m}_2 + \nabla^2 \vec{M}_3$, 其中 \vec{I} 是二阶度规张量， T_1 表示转置， T_2 表示转置第一个和第二个基矢量，认为 \vec{m}_2 和 \vec{M}_3 可由此方程解出。

$$\begin{aligned} &(\vec{v} + \vec{M}_2 \odot_1 \vec{v}) \cdot \nabla (\vec{M}_2 \odot_1 \vec{v}) = \vec{v} \cdot \nabla \vec{M}_2 \cdot \vec{v} + (\vec{M}_2 \cdot \vec{v}) \cdot (\nabla \vec{M}_2 \cdot \vec{v}) \\ &= (\vec{I} + \vec{M}_2) \cdot \vec{v} \cdot \nabla \vec{M}_2 \cdot \vec{v} \\ &= (\vec{I} + \vec{M}_2)^{ij} v_j \vec{g}_i \cdot (\nabla \vec{M}_2)_{mkl} \vec{g}^m \vec{g}^k \vec{g}^l \cdot \vec{v} \\ &= (\vec{I} + \vec{M}_2)^{ij} v_j (\nabla \vec{M}_2)_{ikl} \vec{g}^k \vec{g}^l \cdot \vec{v} \\ &= (\vec{I} + \vec{M}_2)^{T,ji} v_j (\nabla \vec{M}_2)_{ikl} \vec{g}^k \vec{g}^l \cdot \vec{v} \\ &= \left[(\vec{I} + \vec{M}_2)^{T_1} \cdot (\nabla \vec{M}_2) \right]_{kl}^j v_j \vec{g}^k \vec{g}^l \cdot \vec{v} \\ &= \left[(\vec{I} + \vec{M}_2)^{T_1} \cdot (\nabla \vec{M}_2) \right]_{jkl}^n \vec{g}^n \vec{g}^k (\vec{g}^j \cdot \vec{v}) (\vec{g}^l \cdot \vec{v}) \\ &= \left[(\vec{I} + \vec{M}_2)^{T_1} \cdot \nabla \vec{M}_2 \right]_{2}^{T_2} \odot_2 \vec{v} \vec{v} \end{aligned}$$

c. 请考察等式两边n次 \vec{v} 的项，并给出关于 \vec{m}_n 和 \vec{M}_{n+1} 的方程。

解：

先把左边拆分成两项，并且由于 \vec{v} 是常矢量，我们有：

$$\begin{aligned} & \left[\vec{v} + \sum_{i=1}^{\infty} \vec{M}_{i+1} \odot_i (\vec{v} \otimes \vec{v}) \right] \cdot \nabla \sum_{i=1}^{\infty} \vec{M}_{i+1} \odot_i (\vec{v} \otimes \vec{v}) \\ &= \vec{v} \cdot \sum_{i=1}^{\infty} \nabla \vec{M}_{i+1} \odot_i (\vec{v} \otimes \vec{v}) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[\vec{M}_{i+1} \odot_i (\vec{v} \otimes \vec{v}) \right] \cdot \left[\nabla \vec{M}_{j+1} \odot_j (\vec{v} \otimes \vec{v}) \right] \end{aligned}$$

故关于 \vec{v} 的n次项为

$$\vec{v} \cdot \nabla \vec{M}_n \odot_{n-1} (\vec{v} \otimes \vec{v}) + \sum_{i=1, j=1}^{i+j=n} \vec{M}_{i+1} \odot_i (\vec{v} \otimes \vec{v}) \cdot \nabla \vec{M}_{j+1} \odot_j (\vec{v} \otimes \vec{v})$$

而右边关于 \vec{v} 的项则为

$$(-\nabla \vec{m}_n + \nabla^2 \vec{M}_{n+1}) \odot_n (\vec{v} \otimes \vec{v})$$

先看左边第一项：

$$\begin{aligned} & \vec{v} \cdot \nabla \vec{M}_n \odot_{n-1} (\vec{v} \otimes \vec{v}) = v^{i_1} (\nabla \vec{M}_n)_{i_1 \dots i_{n+1}} \vec{g}^{i_1} \dots \vec{g}^{i_{n+1}} \odot_{n-1} (\vec{v} \otimes \vec{v}) \\ &= (\nabla \vec{M}_n)_{i_1 i_2 (i_3 i_4 \dots i_{n+1})} v^{i_1} \vec{g}^{i_2} \vec{g}^{i_3} \dots \vec{g}^{i_{n+1}} \odot_{n-1} (\vec{v} \otimes \vec{v}) \\ &= (\nabla \vec{M}_n)_{i_1 i_2 (i_3 i_4 \dots i_{n+2})} \vec{g}^{i_2} (\vec{g}^{i_1} \cdot \vec{v}) \dots \vec{g}^{i_{n+2}} \odot_{n-1} (\vec{v} \otimes \vec{v}) \\ &= (\nabla \vec{M}_n)_{i_2 (i_1 i_3 i_4 \dots i_{n+2})} \vec{g}^{i_2} (\vec{g}^{i_1} \vec{g}^{i_3} \vec{g}^{i_4} \dots \vec{g}^{i_{n+2}}) \odot_n (\vec{v} \otimes \vec{v}) \end{aligned}$$

第二项： $(i+j) = n$

$$\begin{aligned} & \left[\vec{M}_{i+1} \odot_i (\vec{v} \otimes \vec{v}) \right] \cdot \left[\nabla \vec{M}_{j+1} \odot_j (\vec{v} \otimes \vec{v}) \right] \\ &= \left[(\vec{M}_{i+1})_{i_1 (i_2 \dots i_{i+1})} \vec{g}^{i_1} (\vec{g}^{i_2} \dots \vec{g}^{i_{i+1}}) \odot_i (\vec{v} \otimes \vec{v}) \right] \cdot \left[(\nabla \vec{M}_{j+1})_{j_1 j_2 (j_3 j_4 \dots j_{j+2})} \vec{g}^{j_1} \dots \vec{g}^{j_{j+2}} \odot_j (\vec{v} \otimes \vec{v}) \right] \\ &= (\vec{M}_{i+1})_{i_1 (i_2 \dots i_{i+1})} (\vec{g}^{i_1} \cdot \vec{g}^{j_1}) (\nabla \vec{M}_{j+1})_{j_1 j_2 (j_3 j_4 \dots j_{j+2})} \vec{g}^{j_2} (\vec{g}^{i_2} \dots \vec{g}^{i_{i+1}}) \odot_i (\vec{v} \otimes \vec{v}) (\vec{g}^{j_3} \vec{g}^{j_4} \dots \vec{g}^{j_{j+2}}) \odot_j (\vec{v} \otimes \vec{v}) \\ &= (\vec{M}_{i+1})_{i_2 \dots i_{i+1} i_1} (\vec{g}^{i_1} \cdot \vec{g}^{j_1}) (\nabla \vec{M}_{j+1})_{j_1 j_2 (j_3 j_4 \dots j_{j+2})} (\vec{g}^{i_2} \dots \vec{g}^{i_{i+1}}) \odot_i (\vec{v} \otimes \vec{v}) (\vec{g}^{j_3} \vec{g}^{j_4} \dots \vec{g}^{j_{j+2}}) \odot_j (\vec{v} \otimes \vec{v}) \\ &= \left[\vec{M}_{i+1}^{T_2 \dots T_{i+1}} \cdot \nabla \vec{M}_{j+1} \right]_{(i_2 \dots i_{i+1}) j_2 (j_3 j_4 \dots j_{j+2})} \vec{g}^{j_2} (\vec{g}^{i_2} \dots \vec{g}^{i_{i+1}}) \odot_i (\vec{v} \otimes \vec{v}) (\vec{g}^{j_3} \vec{g}^{j_4} \dots \vec{g}^{j_{j+2}}) \odot_j (\vec{v} \otimes \vec{v}) \\ &= \left[\vec{M}_{i+1}^{T_2 \dots T_{i+1}} \cdot \nabla \vec{M}_{j+1} \right]_{j_2 (i_2 \dots i_{i+1}) (j_3 j_4 \dots j_{j+2})}^{S_{i+1}} (\vec{g}^{i_2} \dots \vec{g}^{i_{i+1}}) \odot_i (\vec{v} \otimes \vec{v}) (\vec{g}^{j_4} \dots \vec{g}^{j_{j+3}}) \odot_j (\vec{v} \otimes \vec{v}) \\ &= \left[\vec{M}_{i+1}^{T_2 \dots T_{i+1}} \cdot \nabla \vec{M}_{j+1} \right]_n^{S_{i+1}} \odot_n (\vec{v} \otimes \vec{v}) \end{aligned}$$

S_{i+1} 表示把第 $i+1$ 个基矢量shift到第一个。 T_i 表示调换第*i*和第*i*-1个基矢量。

故最后的表示式为：

$$(\nabla \vec{M}_n)^{T_2} + \sum_{i=1}^{i+j=n} \left[\vec{M}_{i+1}^{T_2 \dots T_{i+1}} \cdot \nabla \vec{M}_{j+1} \right]^{S_{i+1}} = -\nabla \vec{m}_n + \nabla^2 \vec{M}_{n+1}$$

$$S_{i+1} = T_{i+1} \dots T_2$$

33. 写出曲面的第一标准张量和第二标准张量；并证明第二标准型等于 $-d\vec{r} \cdot d\vec{n}$ 。

解：

(此处 \vec{p} 即 \vec{r})

第一标准张量： $\vec{a} = a_{\alpha\beta} \vec{p}^\alpha \vec{p}^\beta$, $a_{\alpha\beta} = \vec{p}_\alpha \cdot \vec{p}_\beta$, \vec{p}_α 为曲面上的自然基矢量。

第二基本张量： $\vec{b} = b_{\alpha\beta} \vec{p}^\alpha \vec{p}^\beta$, $b_{\alpha\beta} = \vec{n} \cdot \frac{\partial^2 \vec{p}}{\partial \xi^\alpha \partial \xi^\beta} = \vec{n} \cdot \frac{\partial \vec{p}_\alpha}{\partial \xi^\beta} = -\frac{\partial \vec{n}}{\partial \xi^\beta} \cdot \vec{p}_\alpha$

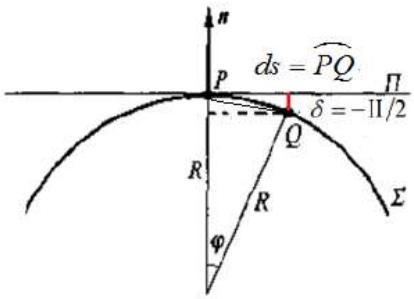
最后一个等式来自于： $\vec{n} \cdot \vec{p}_\alpha = 0 \Rightarrow \vec{n} \cdot \frac{\partial \vec{p}_\alpha}{\partial \xi^\beta} + \frac{\partial \vec{n}}{\partial \xi^\beta} \cdot \vec{p}_\alpha = 0$

故有：

$$II = b_{\alpha\beta} d\xi^\alpha d\xi^\beta = -\frac{\partial \vec{n}}{\partial \xi^\beta} \cdot \frac{\partial \vec{p}}{\partial \xi^\alpha} d\xi^\alpha d\xi^\beta = -\left(\frac{\partial \vec{n}}{\partial \xi^\beta} d\xi^\beta \right) \cdot \left(\frac{\partial \vec{p}}{\partial \xi^\alpha} d\xi^\alpha \right) = -d\vec{n} \cdot d\vec{p} = -d\vec{r} \cdot d\vec{n}$$

34. 证明：二维曲面的Guass曲率 K 等于曲面第二基本张量 \vec{b} 行列式的值。

解：



Guass曲率为两个主曲率之积: $K = \kappa_1 \kappa_2$

主曲率是法截面曲率 κ 的极值。

对于如图所示的曲面, 有:

$$\begin{aligned}\kappa &= \lim_{\Delta s \rightarrow 0} \frac{\phi}{\Delta s} = \frac{1}{R} = \frac{(ds)^2}{R(ds)^2} \\ ds &\approx \overline{PQ} = \frac{\delta}{\cos\left(\frac{\pi}{2} - \frac{\phi}{2}\right)} \approx \frac{2\delta}{\phi} \approx \frac{2\delta R}{ds} \\ \therefore \kappa &= \frac{2\delta}{(ds)^2} = -\frac{II}{I} = -\frac{b_{\alpha\beta} d\xi^\alpha d\xi^\beta}{a_{\lambda\gamma} d\xi^\lambda d\xi^\gamma}\end{aligned}$$

对于法截面曲线的切向单位矢量: $\vec{t} = \frac{d\vec{r}}{ds} = \vec{r}_\alpha \frac{d\xi^\alpha}{ds} = t^\alpha \vec{r}_\alpha$

$$\text{则: } \kappa = -\frac{II}{I} = -\frac{b_{\alpha\beta} d\xi^\alpha d\xi^\beta}{a_{\lambda\gamma} d\xi^\lambda d\xi^\gamma} = -b_{\alpha\beta} \left(\frac{d\xi^\alpha}{ds} \right) \left(\frac{d\xi^\beta}{ds} \right) = -b_{\alpha\beta} t^\alpha t^\beta$$

\vec{t} 满足约束条件: $\vec{t} \cdot \vec{t} = 1 \Rightarrow \text{Lagrange 乘子法:}$

$$\frac{\partial}{\partial t^\omega} [-b_{\alpha\beta} t^\alpha t^\beta + \lambda(a_{\alpha\beta} t^\alpha t^\beta - 1)] = 0 \Rightarrow -b_{\omega\alpha} t^\alpha + \lambda a_{\omega\alpha} t^\alpha = 0$$

将 ω 升高, 可以得到 $(b_\alpha^\omega - \lambda \delta_{\alpha\omega}^\omega) t^\alpha = 0$

故 κ 取极值时, \vec{t} 对应着 b 张量的特征矢, λ 为其特征值。

由于其对称正定, 其必存在两个实特征值 λ_1, λ_2

而此时: $\kappa_1 = -\vec{t}_1 \cdot \vec{b} \cdot \vec{t}_1 = -\lambda_1 (\vec{t}_1 \cdot \vec{t}_1) = -\lambda_1$, 同理 $\kappa_2 = -\lambda_2$

故 $K = \kappa_1 \kappa_2 = \lambda_1 \lambda_2 = \det(b_\alpha^\omega)$

35. 写出二维曲面上三阶张量 \vec{T} 的分量形式表示。

解:

$$\begin{aligned}\vec{T} &= T_{\alpha\beta\gamma} \vec{r}^\alpha \vec{r}^\beta \vec{r}^\gamma + T_{n\alpha\beta} \vec{n} \vec{r}^\alpha \vec{r}^\beta + T_{\alpha n\beta} \vec{r}^\alpha \vec{n} \vec{r}^\beta + T_{\alpha\beta n} \vec{r}^\alpha \vec{r}^\beta \vec{n} \\ &+ T_{nn\alpha} \vec{n} \vec{n} \vec{r}^\alpha + T_{n\alpha n} \vec{n} \vec{r}^\alpha \vec{n} + T_{\alpha n n} \vec{r}^\alpha \vec{n} \vec{n} + T_{nnn} \vec{n} \vec{n} \vec{n}\end{aligned}$$

36. 计算闭合圆环曲面上的Riemann-Christoffel张量, 曲面坐标 (θ, ϕ) 与笛卡尔坐标的关系为: $\begin{cases} x = (R + r_0 \sin \theta) \cos \phi \\ y = (R + r_0 \sin \theta) \sin \phi \\ z = r_0 \cos \theta \end{cases}$

解:

$$\vec{g}_\theta = \frac{\partial \vec{r}}{\partial \theta} = r_0 \cos \theta \cos \phi \vec{i} + r_0 \cos \theta \sin \phi \vec{j} - r_0 \sin \theta \vec{k}$$

$$\vec{g}_\phi = \frac{\partial \vec{r}}{\partial \phi} = -(R + r_0 \sin \theta) \sin \phi \vec{i} + (R + r_0 \sin \theta) \cos \phi \vec{j}$$

$$\vec{g}^\theta = \frac{1}{r_0} \cos \theta \cos \phi \vec{i} + \frac{1}{r_0} \cos \theta \sin \phi \vec{j} - \frac{1}{r_0} \sin \theta \vec{k}$$

$$\vec{g}^\phi = -\frac{1}{(R + r_0 \sin \theta)} \sin \phi \vec{i} + \frac{1}{(R + r_0 \sin \theta)} \cos \phi \vec{j}$$

$$\vec{n} = \sin \theta \cos \phi \vec{i} + \sin \theta \sin \phi \vec{j} + \cos \theta \vec{k}$$

$$[g_{ij}] = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\theta\phi} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} r_0^2 & 0 \\ 0 & (R + r_0 \sin \theta)^2 \end{bmatrix}$$

$$[g^{ij}] = \begin{bmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\theta\phi} & g^{\phi\phi} \end{bmatrix} = \begin{bmatrix} \frac{1}{r_0^2} & 0 \\ 0 & \frac{1}{(R + r_0 \sin \theta)^2} \end{bmatrix}$$

$$\Gamma_{ij}^\theta = \frac{1}{2} g^{\theta\theta} \left[\frac{\partial g_{i\theta}}{\partial x^j} + \frac{\partial g_{j\theta}}{\partial x^i} - \frac{\partial g_{ij}}{\partial \theta} \right]$$

$$\Rightarrow \begin{cases} \Gamma_{\theta\theta}^\theta = \frac{1}{2r_0^2}[0+0-0] = 0 \\ \Gamma_{\theta\phi}^\theta = \frac{1}{2r_0^2}\left[\frac{\partial g_{\theta\theta}}{\partial\phi} + \frac{\partial g_{\phi\theta}}{\partial\theta} - \frac{\partial g_{\theta\phi}}{\partial\theta}\right] = 0 \\ \Gamma_{\phi\phi}^\theta = \frac{1}{2r_0^2}\left[\frac{\partial g_{\phi\theta}}{\partial\phi} + \frac{\partial g_{\phi\theta}}{\partial\phi} - \frac{\partial g_{\phi\phi}}{\partial\theta}\right] = -\frac{(R+r_0 \sin\theta)\cos\theta}{r_0} \\ \Gamma_{ij}^\phi = \frac{1}{2}g^{\phi\phi}\left[\frac{\partial g_{i\phi}}{\partial x^j} + \frac{\partial g_{j\phi}}{\partial x^i} - \frac{\partial g_{ij}}{\partial\phi}\right] \\ \Gamma_{\phi\phi}^\phi = \frac{1}{2(R+r_0 \sin\theta)^2}[0+0-0] = 0 \\ \Gamma_{\theta\phi}^\phi = \frac{1}{2(R+r_0 \sin\theta)^2}\left[\frac{\partial g_{\theta\phi}}{\partial\phi} + \frac{\partial g_{\phi\phi}}{\partial\theta} - \frac{\partial g_{\theta\phi}}{\partial\phi}\right] = \frac{r_0 \cos\theta}{R+r_0 \sin\theta} \\ \Gamma_{\theta\theta}^\phi = \frac{1}{2(R+r_0 \sin\theta)^2}\left[\frac{\partial g_{\theta\phi}}{\partial\theta} + \frac{\partial g_{\theta\phi}}{\partial\theta} - \frac{\partial g_{\theta\theta}}{\partial\phi}\right] = 0 \end{cases}$$

我们只需计算：

$$\begin{aligned} R_{\phi\theta\phi} &= \frac{\partial\Gamma_{\phi\phi}^\theta}{\partial\theta} - \frac{\partial\Gamma_{\phi\theta}^\theta}{\partial\phi} + \Gamma_{\phi\phi}^\theta\Gamma_{\theta\theta}^\theta - \Gamma_{\phi\theta}^\theta\Gamma_{\phi\phi}^\theta \\ &= \frac{R\sin\theta - r_0\cos2\theta}{r_0} - \frac{r_0\cos\theta}{R+r_0\sin\theta}\left(-\frac{(R+r_0\sin\theta)\cos\theta}{r_0}\right) \\ &= \frac{R}{r_0}\sin\theta - \cos2\theta + \cos^2\theta = \frac{R}{r_0}\sin\theta + \sin^2\theta \\ R_{\theta\phi\theta\phi} &= g_{\theta\theta}R_{\phi\theta\phi}^\theta = Rr_0\sin\theta + r_0^2\sin^2\theta \\ \vec{R} &= r_0\sin\theta(R+r_0\sin\theta)(\vec{g}^\theta\vec{g}^\phi\vec{g}^\theta\vec{g}^\phi - \vec{g}^\theta\vec{g}^\phi\vec{g}^\phi\vec{g}^\theta - \vec{g}^\phi\vec{g}^\theta\vec{g}^\theta\vec{g}^\phi + \vec{g}^\phi\vec{g}^\theta\vec{g}^\phi\vec{g}^\theta) \end{aligned}$$

37. 计算球面上的Riemann-Christoffel张量的分量 R_{1212} ；球面高斯坐标 (θ, ϕ) 与笛卡尔坐标的对应关系为： $\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}$

$$(\Gamma_{ij}^p = \frac{1}{2}g^{kp}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right), R_{ijkl} = g_{ir}\left(\frac{\partial\Gamma_{jl}^r}{\partial x^k} - \frac{\partial\Gamma_{jk}^r}{\partial x^l} + \Gamma_{sk}^r\Gamma_{jl}^s - \Gamma_{jk}^s\Gamma_{sl}^r\right))$$

解：

利用几何法可以得到球面上的线元：

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2$$

故可以得到其度量张量为：

$$g_{11} = R^2, g_{22} = R^2 \sin^2\theta \Rightarrow g^{11} = \frac{1}{R^2}, g^{22} = \frac{1}{R^2 \sin^2\theta}$$

故马上得到 $\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0$

$$\Gamma_{12}^2 = \cot\theta, \Gamma_{22}^1 = -\frac{1}{2}\sin2\theta$$

故：

$$\begin{aligned} R_{1212} &= g_{11}\left(\frac{\partial\Gamma_{22}^1}{\partial\theta} - 0 + 0 + 0 - 0 - \Gamma_{12}^2\Gamma_{22}^1\right) \\ &= R^2\left(-\cos2\theta + \frac{1}{2}\sin2\theta\cot\theta\right) = R^2\sin^2\theta \\ \text{故 } \vec{R} &= R^2\sin^2\theta(\vec{g}^\theta\vec{g}^\phi\vec{g}^\theta\vec{g}^\phi - \vec{g}^\theta\vec{g}^\phi\vec{g}^\phi\vec{g}^\theta - \vec{g}^\phi\vec{g}^\theta\vec{g}^\theta\vec{g}^\phi + \vec{g}^\phi\vec{g}^\theta\vec{g}^\phi\vec{g}^\theta) \end{aligned}$$

38. 试用速度场 \vec{V} 和拉格朗日自然基矢量 \hat{g}_i 表示 $\frac{\partial\hat{g}_i}{\partial t}$ 。

解：

$$\hat{g}_i = \left(\frac{\partial\vec{r}}{\partial\xi^i}\right)_t = \left(\frac{\partial x^j}{\partial\xi^i}\right)_t \vec{g}_j$$

则：

$$\left(\frac{\partial\hat{g}_i}{\partial t}\right)_{x^j} = \left(\frac{\partial\hat{g}_i(\xi^k(x^j, t), t)}{\partial t}\right)_{x^j} = \left(\frac{\partial\hat{g}_i}{\partial\xi^k}\right)_t \left(\frac{\partial\xi^k}{\partial t}\right)_{x^j} + \left(\frac{\partial\hat{g}_i}{\partial t}\right)_{\xi^k}$$

$$\left(\frac{\partial\hat{g}_i}{\partial t}\right)_{\xi^k} = \frac{d\hat{g}_i}{dt} = \frac{\partial^2\vec{r}}{\partial t\partial\xi^i} = (\vec{V}\nabla)\cdot\hat{g}_i = (\vec{V}\nabla)\cdot\hat{g}_i$$

$$\left(\frac{\partial\hat{g}_i}{\partial\xi^k}\right)_t \left(\frac{\partial\xi^k}{\partial t}\right)_{x^j} = \left(\frac{\partial\xi^k}{\partial t}\right)_{x^j} \left(\frac{\partial\hat{g}_i}{\partial\xi^k}\right)_t$$

$$\text{利用: } \left(\frac{\partial\xi^k}{\partial t}\right)_{x^j} \left(\frac{\partial t}{\partial x^j}\right)_{\xi^k} \left(\frac{\partial x^j}{\partial\xi^k}\right)_t = -1 \Rightarrow \left(\frac{\partial\xi^k}{\partial t}\right)_{x^j} = -\left(\frac{\partial x^j}{\partial\xi^k}\right)v^j$$

$$\begin{aligned} \left(\frac{\partial \hat{g}_i}{\partial \xi^k} \right)_t \left(\frac{\partial \xi^k}{\partial t} \right)_{x^j} &= - \left(\frac{\partial \xi^k}{\partial x^l} \right) v^j (\vec{g}_j \cdot \vec{g}^l) \left(\frac{\partial \hat{g}_i}{\partial \xi^k} \right)_t \\ &= - \vec{V} \cdot \left(\frac{\partial \xi^k}{\partial x^l} \right) \vec{g}^l \left(\frac{\partial \hat{g}_i}{\partial \xi^k} \right)_t = - \vec{V} \cdot \vec{g}^k \left(\frac{\partial \hat{g}_i}{\partial \xi^k} \right)_t = - \vec{V} \cdot \nabla \hat{g}_i \end{aligned}$$

或注意到:

$$\frac{\partial \hat{g}_i}{\partial t} = \frac{\partial \hat{g}_i}{\partial t^\xi} - \vec{V} \cdot \nabla \hat{g}_i = (\vec{V} \nabla) \cdot \hat{g}_i - \vec{V} \cdot \nabla \hat{g}_i$$

39. 简述拉格朗日坐标系和欧拉坐标系的区别; 并证明物质导数公式:

$$\frac{\partial \vec{T}}{\partial t^\xi} = \frac{\partial \vec{T}}{\partial t} + (V \cdot \nabla) \vec{T}$$

解:

Lagrange 坐标为嵌在物体质点上的, 随着物体一起运动和变形的目标, 又叫随体目标, 记作 ξ^i , 自然基矢量可能时有关。

Euler 坐标的 x^i 是固定在空间中的参考坐标, 又称空间坐标, 自然基矢量时无关。

$$\frac{\partial \vec{T}}{\partial t^\xi} = \frac{d \vec{T}}{dt} = \left(\frac{\partial \vec{T}}{\partial t} \right)_{x^k} + \left(\frac{\partial \vec{T}}{\partial x^k} \right)_t \left(\frac{\partial x^k}{\partial t} \right)_{\xi^i} = \frac{\partial \vec{T}}{\partial t} + v^k \frac{\partial \vec{T}}{\partial x^k} = \frac{\partial \vec{T}}{\partial t} + (\vec{V} \cdot \nabla) \vec{T}$$

40. 有一个静止的笛卡尔坐标系 $(\vec{i}, \vec{j}, \vec{k})$, 另一个笛卡尔坐标系 $(\hat{i}, \hat{j}, \hat{k})$ 以角速度 $\vec{\omega}$ 旋转, 求坐标变换关系 $\hat{x} = \hat{x}(x, y, z, t)$ 。
(假设原点重合。)

解:

$$\begin{cases} \frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i} \\ \frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j} \\ \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k} \end{cases} \Rightarrow \begin{cases} \frac{d\hat{i}_x}{dt} = \omega_y \hat{i}_z - \omega_z \hat{i}_y \\ \frac{d\hat{i}_y}{dt} = \omega_z \hat{i}_x - \omega_x \hat{i}_z \\ \frac{d\hat{i}_z}{dt} = \omega_x \hat{i}_y - \omega_y \hat{i}_x \end{cases} ; \begin{cases} \frac{d\hat{j}_x}{dt} = \omega_y \hat{j}_z - \omega_z \hat{j}_y \\ \frac{d\hat{j}_y}{dt} = \omega_z \hat{j}_x - \omega_x \hat{j}_z \\ \frac{d\hat{j}_z}{dt} = \omega_x \hat{j}_y - \omega_y \hat{j}_x \end{cases} ; \begin{cases} \frac{d\hat{k}_x}{dt} = \omega_y \hat{k}_z - \omega_z \hat{k}_y \\ \frac{d\hat{k}_y}{dt} = \omega_z \hat{k}_x - \omega_x \hat{k}_z \\ \frac{d\hat{k}_z}{dt} = \omega_x \hat{k}_y - \omega_y \hat{k}_x \end{cases}$$

$$\text{对 } \hat{i} \text{ 求解有: 令 } \begin{pmatrix} \hat{i}_x \\ \hat{i}_y \\ \hat{i}_z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} e^{\lambda t} \Rightarrow \begin{pmatrix} \lambda & \omega_z & -\omega_y \\ -\omega_z & \lambda & \omega_x \\ \omega_y & -\omega_x & \lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$$

$$\text{则可求得特征值: } \begin{cases} \lambda_1 = 0 \\ \lambda_2 = \omega_i \\ \lambda_3 = -\omega_i \end{cases}$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_1 = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} 1 & \omega_z & -\omega_y \\ \omega_x & 1 & \omega_x \\ \omega_y & -\omega_x & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_3 = \begin{pmatrix} 1 & -\omega_z & \omega_y \\ \omega_x & -\omega_y & -\omega_x \\ \omega_y & \omega_x & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\text{又 } \begin{pmatrix} \hat{i}_x \\ \hat{i}_y \\ \hat{i}_z \end{pmatrix}_{t=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{i}_x \\ \hat{i}_y \\ \hat{i}_z \end{pmatrix} = \frac{1}{\omega^2} \begin{pmatrix} \omega_x^2 + (\omega_y^2 + \omega_z^2) \cos \omega t \\ \omega_x \omega_y (1 - \cos \omega t) + \omega_z \omega \sin \omega t \\ \omega_x \omega_z (1 - \cos \omega t) - \omega_y \omega \sin \omega t \end{pmatrix}$$

轮转即可得到

$$\begin{pmatrix} \hat{i}_x \\ \hat{i}_y \\ \hat{i}_z \end{pmatrix} = \frac{1}{\omega^2} \begin{pmatrix} \omega_x \omega_y (1 - \cos \omega t) - \omega_z \omega \sin \omega t \\ \omega_y^2 + (\omega_x^2 + \omega_z^2) \cos \omega t \\ \omega_y \omega_z (1 - \cos \omega t) + \omega_x \omega \sin \omega t \end{pmatrix}$$

$$\begin{pmatrix} \hat{k}_x \\ \hat{k}_y \\ \hat{k}_z \end{pmatrix} = \frac{1}{\omega^2} \begin{pmatrix} \omega_z \omega_x (1 - \cos \omega t) + \omega_y \omega \sin \omega t \\ \omega_z \omega_y (1 - \cos \omega t) - \omega_x \omega \sin \omega t \\ \omega_z^2 + (\omega_x^2 + \omega_y^2) \cos \omega t \end{pmatrix}$$

$$\text{又 } \hat{x}\hat{i} + \hat{y}\hat{j} + \hat{z}\hat{k} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \begin{pmatrix} \hat{i}_x & \hat{j}_x & \hat{k}_x \\ \hat{i}_y & \hat{j}_y & \hat{k}_y \\ \hat{i}_z & \hat{j}_z & \hat{k}_z \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{可得: } \begin{pmatrix} \hat{i}_x & \hat{j}_x & \hat{k}_x \\ \hat{i}_y & \hat{j}_y & \hat{k}_y \\ \hat{i}_z & \hat{j}_z & \hat{k}_z \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$$\begin{cases} \hat{x} = \frac{1}{\omega^2} [(\omega_x^2 + (\omega_y^2 + \omega_z^2) \cos \omega t) x + (\omega_x \omega_y (1 - \cos \omega t) + \omega_z \omega \sin \omega t) y + (\omega_x \omega_z (1 - \cos \omega t) - \omega_y \omega \sin \omega t) z] \\ \hat{y} = \frac{1}{\omega^2} [(\omega_y^2 + (\omega_x^2 + \omega_z^2) \cos \omega t) y + (\omega_x \omega_y (1 - \cos \omega t) - \omega_z \omega \sin \omega t) x + (\omega_y \omega_z (1 - \cos \omega t) + \omega_x \omega \sin \omega t) z] \\ \hat{z} = \frac{1}{\omega^2} [(\omega_z^2 + (\omega_y^2 + \omega_x^2) \cos \omega t) z + (\omega_x \omega_z (1 - \cos \omega t) - \omega_y \omega \sin \omega t) y + (\omega_x \omega_z (1 - \cos \omega t) + \omega_y \omega \sin \omega t) x] \end{cases}$$

41. 对点源诱导的平面流动，有如下速度场： $\begin{cases} V_r = \frac{Q}{2\pi r} \\ V_\theta = 0 \end{cases}$ ，试写出欧拉坐标和拉格朗日坐标的变换关系；并求变形梯度张量

\vec{F} 、应变率张量 \vec{d} 、Green 应变张量 \vec{E} ，并验证 $F^{-T} \cdot \frac{\partial \vec{E}}{\partial t^\xi} \cdot \vec{F}^{-1} = \vec{d}$ 。

解：

设欧拉坐标和拉格朗日坐标在 $t = 0$ 时重合 $(r(0), \theta(0)) = (r_0, \theta_0)$ ，则有

$$\begin{cases} \frac{dr}{dt} = V_r = \frac{Q}{2\pi r} \\ \frac{d\theta}{dt} = V_\theta = 0 \end{cases} \Rightarrow \begin{cases} r = \sqrt{r_0^2 + \frac{Q}{\pi} t} \\ \theta = \theta_0 \end{cases}$$

(r_0, θ_0) 为拉格朗日坐标。

注意： $\vec{g}^r = \vec{g}_r; \vec{g}^\theta = \frac{1}{r^2} \vec{g}_\theta$

则拉格朗日基矢量：

$$\begin{cases} \hat{\vec{g}}_r = \frac{\partial r}{\partial r_0} \vec{g}_r + \frac{\partial \theta}{\partial r_0} \vec{g}_\theta = \frac{r_0 \vec{g}_r}{\sqrt{r_0^2 + \frac{Q}{\pi} t}} \\ \hat{\vec{g}}_\theta = \frac{\partial r}{\partial \theta_0} \vec{g}_r + \frac{\partial \theta}{\partial \theta_0} \vec{g}_\theta = \vec{g}_\theta \end{cases}$$

故应变率张量：

$$\begin{aligned} \nabla \vec{v} &= \vec{g}^r \left(\frac{\partial v_r}{\partial r} \vec{g}_r + v_r \frac{\partial \vec{g}_r}{\partial r} + \frac{\partial v_\theta}{\partial r} \vec{g}_\theta + v_\theta \frac{\partial \vec{g}_\theta}{\partial r} \right) + \vec{g}^\theta \left(\frac{\partial v_r}{\partial \theta} \vec{g}_r + v_r \frac{\partial \vec{g}_r}{\partial \theta} + \frac{\partial v_\theta}{\partial \theta} \vec{g}_\theta + v_\theta \frac{\partial \vec{g}_\theta}{\partial \theta} \right) \\ &= \vec{g}^r \left(-\frac{Q}{2\pi r^2} \vec{g}_r \right) + \vec{g}^\theta \left(\frac{Q}{2\pi r} \vec{g}_\theta \right) = \frac{Q}{2\pi r^2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} (\vec{g}^i \vec{g}_j) \\ \vec{d} &= \frac{1}{2} (\vec{v} \nabla + \nabla \vec{v}) = \frac{Q}{2\pi r^2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} (\vec{g}^i \vec{g}_j) = \frac{Q}{2\pi r^2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} (\vec{g}_i \vec{g}^j) \end{aligned}$$

而 Green 应变张量：

$$[\hat{g}_{ij}] = \begin{bmatrix} \frac{r_0^2}{r_0^2 + \frac{Q}{\pi} t} & 0 \\ 0 & \frac{r_0^2}{r_0^2 + \frac{Q}{\pi} t} \end{bmatrix}$$

$$\vec{E} = \frac{1}{2} (\hat{g}_{ij} - \frac{\partial}{\partial r_0} \hat{g}_{ij}) \hat{\vec{g}}^i \hat{\vec{g}}^j = \frac{1}{2} \begin{bmatrix} \frac{Q}{\pi} t & 0 \\ r_0^2 + \frac{Q}{\pi} t & 0 \\ 0 & \frac{Q}{\pi} t \end{bmatrix} (\hat{\vec{g}}^i \hat{\vec{g}}^j)$$

$$\begin{aligned} \frac{\partial \vec{E}}{\partial t^\xi} &= \frac{d \vec{E}}{dt} = \frac{1}{2} \begin{bmatrix} -\frac{Q}{\pi} r_0^2 & 0 \\ \left(r_0^2 + \frac{Q}{\pi} t\right)^2 & 0 \\ 0 & Q \end{bmatrix} (\vec{g}^i \vec{g}^j) = \frac{1}{2} \begin{bmatrix} -\frac{Q r_0^2}{\pi r^4} & 0 \\ 0 & \frac{Q}{\pi r^2} \end{bmatrix} (\vec{g}_i \vec{g}^j) \\ &= \frac{Q}{2\pi r^2} \begin{bmatrix} -\frac{r_0^2}{r^2} & 0 \\ 0 & 1 \end{bmatrix} (\vec{g}_i \vec{g}^j) \end{aligned}$$

变形梯度矢量：

$$\vec{F} = \hat{\vec{e}}_i \vec{e}^i = \begin{bmatrix} \frac{r_0}{\sqrt{r_0^2 + \frac{Q}{\pi} t}} & 0 \\ \sqrt{r_0^2 + \frac{Q}{\pi} t} & 1 \end{bmatrix} (\vec{g}_i \vec{g}^j)$$

$$\vec{F}^T = \begin{bmatrix} \frac{r_0}{\sqrt{r_0^2 + \frac{Q}{\pi} t}} & 0 \\ \sqrt{r_0^2 + \frac{Q}{\pi} t} & 1 \end{bmatrix} (\vec{g}_i \vec{g}^j)$$

故即验证：

$$\vec{F}^T \cdot \vec{d} \cdot \vec{F} = \begin{bmatrix} r_0 & 0 \\ r & 1 \end{bmatrix} \frac{Q}{2\pi r^2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_0 & 0 \\ r & 1 \end{bmatrix} = \frac{Q}{2\pi r^2} \begin{bmatrix} -\frac{r_0^2}{r^2} & 0 \\ 0 & 1 \end{bmatrix} = \vec{E}$$

42. 试通过 $|d\hat{\vec{r}}|^2 - |d\vec{r}|^2 = 2d\vec{r} \cdot \vec{E} \cdot d\vec{r}$, 证明应变相关公式 $\frac{\partial \vec{E}}{\partial t^\xi} = \vec{F}^T \cdot \vec{d} \cdot \vec{F}$

解:

$$\begin{aligned} |d\hat{\vec{r}}|^2 - |d\vec{r}|^2 &= (\hat{g}_{ij} - g_{ij}) d\xi^i d\xi^j = 2d\xi^l \hat{g}_l \cdot \hat{g}^i \frac{1}{2} (\hat{g}_{ij} - g_{ij}) \hat{g}^j \cdot d\xi^m \hat{g}_m \\ &= 2d\vec{r} \cdot \frac{1}{2} (\hat{g}_{ij} - g_{ij}) \hat{g}^i \hat{g}^j \cdot d\vec{r} = 2d\vec{r} \cdot \vec{E} \cdot d\vec{r} \\ \therefore \vec{E} &= \frac{1}{2} (\hat{g}_{ij} - g_{ij}) \hat{g}^i \hat{g}^j \\ \frac{\partial \vec{E}}{\partial t^\xi} &= \frac{1}{2} \frac{d\hat{g}_{ij}}{dt} \hat{g}^i \hat{g}^j \end{aligned}$$

而注意到:

$$\begin{aligned} \vec{d} &= \frac{1}{2} \frac{d\hat{g}_{ij}}{dt} \hat{g}^i \hat{g}^j; \vec{F} = \hat{g}_i \hat{g}^i; \vec{F}^T = \hat{g}^i \hat{g}_i \\ \vec{F}^T \cdot \vec{d} \cdot \vec{F} &= \hat{g}^k \hat{g}_k \cdot \frac{1}{2} \frac{d\hat{g}_{ij}}{dt} \hat{g}^i \hat{g}^j \cdot \hat{g}_i \hat{g}^l = \frac{1}{2} \frac{d\hat{g}_{ij}}{dt} \hat{g}^i \hat{g}^j = \frac{\partial \vec{E}}{\partial t^\xi} \end{aligned}$$

43. 试说明对应变 ε_{ij} 的定义1: $d\hat{\vec{r}} \cdot d\hat{\vec{r}} - d\vec{r} \cdot d\vec{r} = 2\varepsilon_{ij} dx^i dx^j$ (x^i 是介质的拉格朗日坐标) 和定义2: $\vec{\varepsilon} = \frac{1}{2}(\vec{u}\nabla + \nabla\vec{u})$ (\vec{u} 是速度场) 是等价的。

解:

$$d\hat{\vec{r}} \cdot d\hat{\vec{r}} - d\vec{r} \cdot d\vec{r} = (\hat{g}_i dx^i) \cdot (\hat{g}_j dx^j) - (\hat{g}_i dx^i) \cdot (\hat{g}_j dx^j) = (\hat{g}_{ij} - g_{ij}) dx^i dx^j$$

$$\text{故有: } \varepsilon_{ij} = \frac{1}{2} (\hat{g}_{ij} - g_{ij})$$

而:

$$\begin{aligned} \frac{1}{2} (\vec{u}\nabla + \nabla\vec{u}) &= \frac{1}{2} (\vec{u}\hat{\nabla} + \hat{\nabla}\vec{u}) = \frac{1}{2} ((\hat{\nabla}_i \hat{u}_k) \hat{g}^k \hat{g}^i + (\hat{\nabla}_k \hat{u}_i) \hat{g}^k \hat{g}^i) \\ &= \frac{1}{2} (\hat{\nabla}_i \hat{u}_k + \hat{\nabla}_k \hat{u}_i) \hat{g}^k \hat{g}^i \\ &= \frac{1}{2} ((\hat{\nabla}_i \hat{u}^j) \hat{g}_{jk} + (\hat{\nabla}_k \hat{u}^j) \hat{g}_{ji}) \hat{g}^k \hat{g}^i \text{ (利用了 } \hat{g}_{jk} \text{ 的协变导数为零)} \\ &= \frac{1}{2} (\hat{g}_k \cdot (\hat{\nabla}_i \hat{u}^j) \hat{g}_j + (\hat{\nabla}_k \hat{u}^j) \hat{g}_j \cdot \hat{g}_i) \hat{g}^k \hat{g}^i \end{aligned}$$

$$\text{但注意到: } (\hat{\nabla}_i \hat{u}^j) \hat{g}_j = \frac{\partial \vec{u}}{\partial x^i} = \frac{\partial}{\partial x^i} \left(\frac{d\hat{\vec{r}}}{dt} \right) = \frac{d}{dt} \left(\frac{\partial \hat{\vec{r}}}{\partial x^i} \right) = \frac{d\hat{g}_i}{dt}$$

$$\text{故上式} = \frac{1}{2} \left(\hat{g}_k \cdot \frac{d\hat{g}_i}{dt} + \frac{d\hat{g}_k}{dt} \cdot \hat{g}_i \right) \hat{g}^k \hat{g}^i = \frac{1}{2} \frac{d\hat{g}_{ki}}{dt} \hat{g}^k \hat{g}^i$$

不妨令 $dt = 1$ (迷惑操作, 参看例4.12与提纲第四章95)

$$\text{上式分量} \approx \frac{1}{2} (\hat{g}_{ij} - g_{ij}) = \varepsilon_{ij}$$