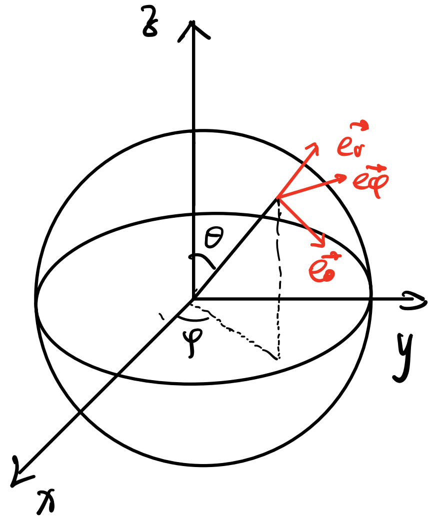
第一次作业 答案



1. $\dot{\vec{r}} = \dot{r}\vec{e_r} + r\dot{\vec{e_r}}$

$\vec{e_r} = \sin\theta\cos\varphi\,\vec{e_x} + \sin\theta\sin\varphi\,\vec{e_y} + \cos\theta\,\vec{e_z}$

$\vec{e_\theta} = \cos\theta\cos\varphi\,\vec{e_x} + \cos\theta\sin\varphi\,\vec{e_y} - \sin\theta\,\vec{e_z}$

$\vec{e_\varphi} = -\sin\varphi\,\vec{e_x} + \cos\varphi\,\vec{e_y}$

$$\dot{\vec{e_r}} = \frac{\partial\vec{e_r}}{\partial r}\dot{r} + \frac{\partial\vec{e_r}}{\partial\theta}\dot{\theta} + \frac{\partial\vec{e_r}}{\partial\varphi}\dot{\varphi}$$

$$= \dot{\theta}\vec{e_\theta} + \sin\theta\dot{\varphi}\vec{e_\varphi}$$

$$\Rightarrow \dot{\vec{r}} = \dot{r}\vec{e_r} + r\dot{\theta}\vec{e_\theta} + r\sin\theta\dot{\varphi}\vec{e_\varphi}$$

$$\ddot{\vec{r}} = \ddot{r}\vec{e_r} + \dot{r}\dot{\vec{e_r}} + \dot{r}\dot{\theta}\vec{e_\theta} + r\ddot{\theta}\vec{e_\theta} + r\dot{\theta}\dot{\vec{e_\theta}} + \dot{r}\sin\theta\dot{\varphi}\vec{e_\varphi}$$
$$+ r\cos\theta\dot{\theta}\dot{\varphi}\vec{e_\varphi} + r\sin\theta\ddot{\varphi}\vec{e_\varphi} + r\sin\theta\dot{\varphi}\dot{\vec{e_\varphi}}$$

$\dot{\vec{e_\theta}} = -\dot{\theta}\vec{e_r} + \cos\theta\dot{\varphi}\vec{e_\varphi}$ $\quad\dot{\vec{e_\varphi}} = -\sin\theta\dot{\varphi}\vec{e_r} - \cos\theta\dot{\varphi}\vec{e_\theta}$

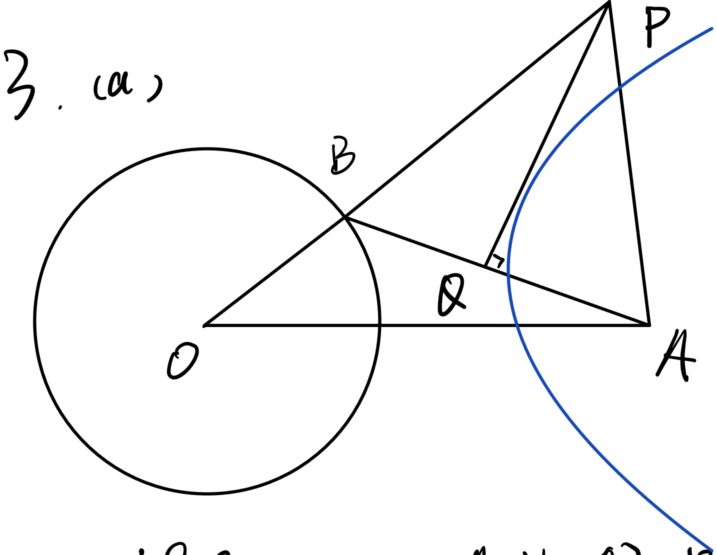
$$\Rightarrow \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi})\vec{e_r}$$
$$+ (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\cos\theta\sin\theta\dot{\varphi}^2)\vec{e_\theta}$$
$$+ (2\dot{r}\dot{\varphi}\sin\theta + 2r\cos\theta\dot{\theta}\dot{\varphi} + r\sin\theta\ddot{\varphi})\vec{e_\varphi}$$

2. $\frac{dv}{dt} = -\frac{GM}{r^2}$ $\quad \frac{dv}{d\theta}\frac{d\theta}{dt} = -\frac{GM}{r^2}$

角动量守恒 $\quad mr^2\dot{\theta} = L = \text{const} \Rightarrow \frac{dv}{d\theta} = \frac{GMm}{L}$

即 $\frac{dv}{d\theta}$ 是常数

3. (a)

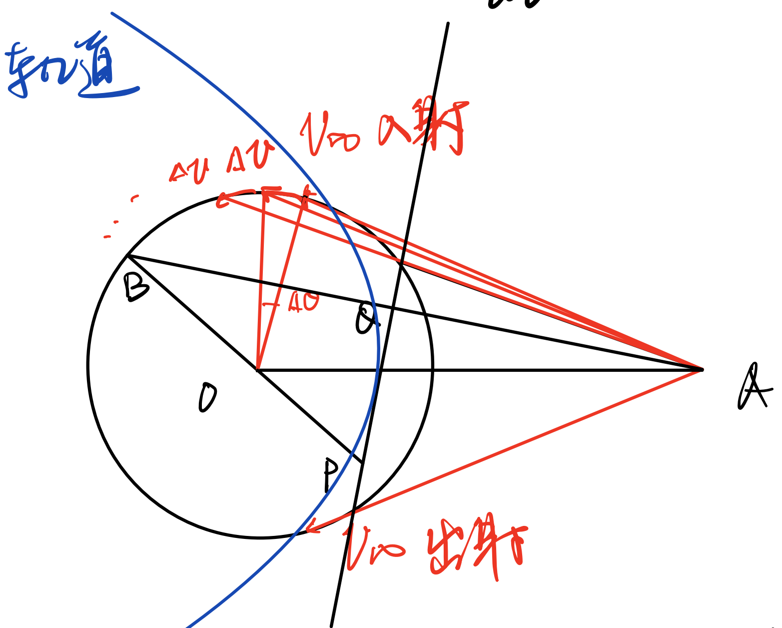


仿照构造椭圆的方法：取圆外点A，圆上点B，作AB中垂线与直线OB交于P。可得 $|OP|-|AP| = R =$ 常数

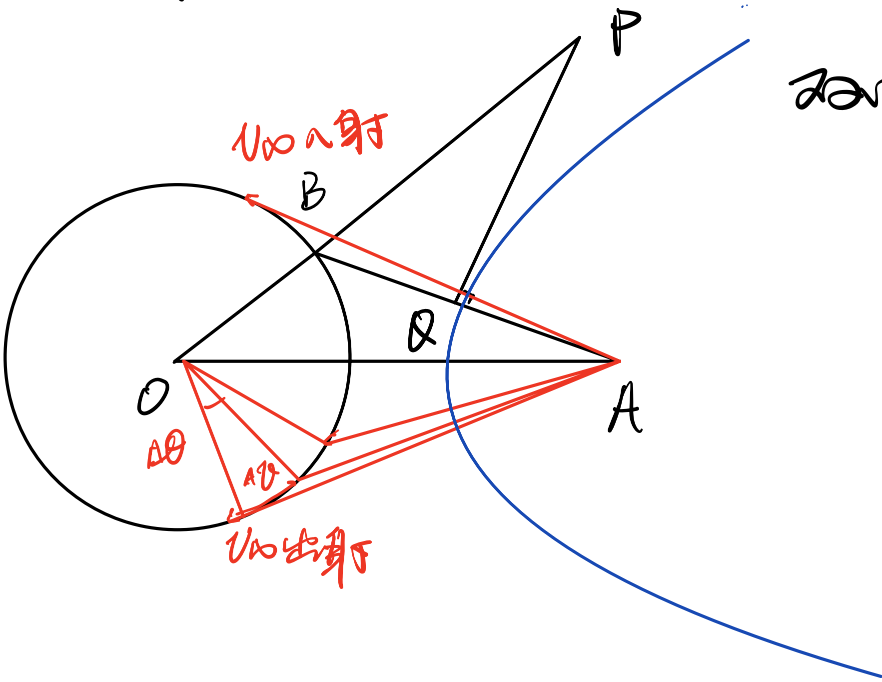
得到以O.A为焦点的双曲线，PQ是切线方向

(b) 由2.可知 $\frac{dv}{d\theta} = const$. 速度图仍为圆的一部分

将速度矢量转过90°，在PQ方向上，此时PQ是双曲线的切线，得到轨道是双曲线"左侧"一支

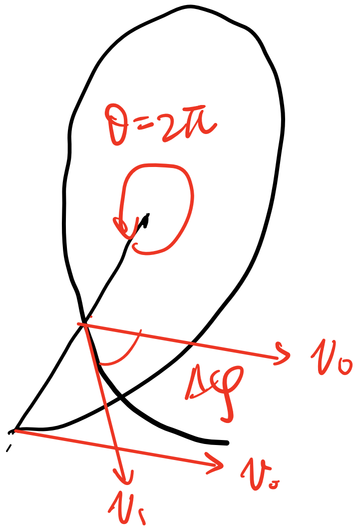


(c) 作用力改为斥力，速度图是劣弧，轨道是双曲线"右侧"一支。



4. 同志们从事科学工作，胆子要大。

$$F = F^{(0)} + F^{(GR)} = \frac{GM_\odot}{r^2} + \frac{3(GM_\odot)^2 p}{r^4 c^2}$$



在速度图上，第一项引起的Δv在圆弧上，对每个Δv，第二项引起一小量$\delta(\Delta v)$，累积角度为2π后，速度方向与初始夹角即为进动角$\Delta\varphi$

进动角 $$\frac{\Delta\varphi}{2\pi} = \frac{\delta(\Delta v)}{\Delta v} \approx \frac{F^{(GR)}}{F^{(0)}} = \frac{3GM_\odot/c^2 p}{r^2} \approx \frac{3GM_\odot/c^2}{p}$$

$$\Delta\varphi = \frac{6\pi(GM_\odot/c^2)}{p} = 5.0\times10^{-7}\ \text{rad}$$

(数量级正确即可)

暴力计算：(梁书8.2)

运动方程 $\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{F}{m}$ 令 $u = \frac{1}{r}$.

得到 $\frac{d^2 u}{d\theta^2} + u = \alpha + \beta u^2$ $\quad F = F^{(0)} + F^{(GR)}$

$\alpha = \frac{GM}{L^2}$ $\quad \beta = \frac{3G^2M^2 p}{mLc^2}$

设 $u=u^{(0)}+u^{(1)}$ $\quad u^{(0)}=\alpha(1+e\cos\theta)$ 迭代

$$\frac{d^2u^{(1)}}{d\theta^2}+u^{(1)}=\alpha+\beta\alpha^2(1+e\cos\theta)^2$$

不难验证 $\quad u^{(1)}=\beta\alpha^2\left(1+\frac{e^2}{2}+e\theta\sin\theta-\frac{e^2}{6}\cos2\theta\right)$

仅有 $e\theta\sin\theta$ 项对进动有贡献

u 可近似作 $\quad u=\alpha(1+e\cos\theta+e\alpha\beta\theta\sin\theta)$

$$=\alpha(1+e\cos(\theta-\alpha\beta\theta))$$

进动角 $\alpha\beta\cdot2\pi=5.0\times10^{-7}\,\text{rad}$.