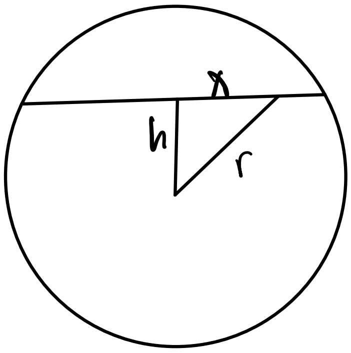
第3次作业

1. (1)



设参量如图, $T=\frac{1}{2}m\dot{x}^2$

$V=\frac{1}{2}m\frac{g}{R}(x^2+h^2) \quad L=T-V$

可得运动方程 $\ddot{x}+\frac{g}{R}x=0$

$\Rightarrow t=\pi\sqrt{\frac{R}{g}}$

(2) 能量守恒 $\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)=\frac{g}{R}(R^2-r^2)$

$(v\,dt)^2=(dr)^2+(r\,d\theta)^2$

$\Rightarrow dt=\sqrt{\frac{R}{g}}\sqrt{\frac{r^2(\frac{d\theta}{dr})^2+1}{R^2-r^2}}\,dr$

最速降线要求 $\delta\left[\int dt\right]=\sqrt{\frac{R}{g}}\,\delta\left[\int\underbrace{\sqrt{\frac{r^2(\frac{d\theta}{dr})^2+1}{R^2-r^2}}}_{f}\,dr\right]=0$

代入Euler方程 $\frac{d}{dr}\frac{\partial f}{\partial\dot{\theta}}-\frac{\partial f}{\partial\theta}=0$

$\Rightarrow \frac{r^2}{\sqrt{R^2-r^2}\sqrt{(\frac{dr}{d\theta})^2+r^2}}=const=\frac{r_0}{\sqrt{R^2-r_0^2}}$

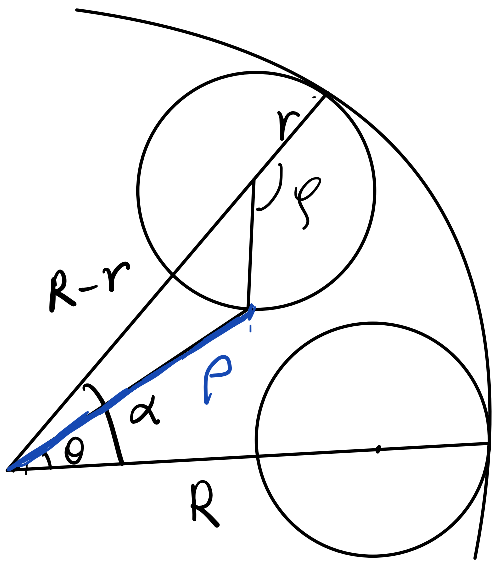
积分可得到θ的表达式

$\theta=\arctan\left(\frac{R}{r_0}\sqrt{\frac{r^2-r_0^2}{R^2-r^2}}\right)-\frac{r_0}{R}\arctan\sqrt{\frac{r^2-r_0^2}{R^2-r^2}}$

令 $\sqrt{\frac{r^2-r_0^2}{R^2-r^2}}=\tan\frac{\varphi}{2} \Rightarrow r^2=\frac{1}{2}(R^2+r_0^2)-\frac{1}{2}(R^2-r_0^2)\cos\varphi$

$\theta=\arctan\left(\frac{R}{r_0}\tan\frac{\varphi}{2}\right)-\frac{\varphi}{2}\frac{r_0}{R}$

(3) 设圆在大圆内滚动，参量如图



$$\rho^2 = (R-r)^2 + r^2 - 2(R-r)r\cos\varphi$$

$$r\varphi = R\alpha$$

$$\frac{r}{\sin(\alpha-\theta)} = \frac{R-r}{\sin(\varphi-(\alpha-\theta))}$$

$$\Rightarrow \theta = \frac{r}{R}\varphi - \arctan\left[\frac{r\sin\varphi}{R+r(\cos\varphi-1)}\right]$$

令 $r = \frac{1}{2}(R-r_0)$ $\quad \rho^2 = \frac{1}{2}(R^2+r_0^2) + \frac{1}{2}(R^2-r_0^2)\cos\varphi$

由 $\tan\left(\theta + \frac{r_0}{2R}\varphi\right) = \frac{r_0}{R}\tan\frac{\varphi}{2}$

$$\theta = \frac{1}{2}\varphi - \arctan\frac{(R-r_0)\sin\varphi}{R+r_0+(R-r_0)\cos\varphi} - \frac{r_0}{2R}\varphi$$

令 $\varphi' = \pi - \varphi$

$$\Rightarrow \rho^2 = \frac{1}{2}(R^2+r_0^2) - \frac{1}{2}(R^2-r_0^2)\cos\varphi'$$

$$\theta = \arctan\left(\frac{R}{r_0}\tan\frac{\varphi'}{2}\right) - \frac{r_0}{2R}\varphi'$$

(4) 可以猜粒子在隧道内运动与匀速滚动的圆上点速度处处相同，验证即可，也可以计算：

$$\frac{d\theta}{dt} = \sqrt{\frac{g}{R}}\sqrt{\frac{R^2-r^2}{\dot{r}^2+r^2}} = \sqrt{\frac{g}{R}}\,\frac{R^2-r^2}{r^2}\,\frac{r_0}{\sqrt{R^2-r_0^2}}$$

又 $$d\theta = \frac{r_0}{2R}\,\frac{(R^2-r_0^2)\cos^2\frac{\varphi}{2}}{R^2\sin^2\frac{\varphi}{2}+r_0^2\cos^2\frac{\varphi}{2}}\,d\varphi$$ （θ参数方程给出）

$$\frac{R^2-r^2}{r^2} = \frac{2(R^2-r_0^2)\cos^2\frac{\varphi}{2}}{R^2+r_0^2-(R^2-r_0^2)\cos\varphi}$$ （r参数方程给出）

积分得到 $t = \frac{\sqrt{R^2-r_0^2}}{2R}\sqrt{\frac{R}{g}}\,\varphi \Rightarrow \varphi = 2\pi\frac{t}{\tau}$

合肥－北京 纬度相差 $9°$

$R\theta = \pi(R-r_0) \Rightarrow r_0 = 0.95R$

$t = 13\,\mathrm{min}$

2. 由 $\sqrt{g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}} = 1$

得 $$\delta s = \frac{1}{2}\int\left(\delta g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)ds + \int g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds}\,ds = 0$$

第二部分，分部积分后

$$= \underbrace{g_{\mu\nu}\frac{dx^\nu}{ds}\delta x^\mu\Big|_i^f}_{=0} - \int\left(g_{\mu\nu}\frac{d^2x^\nu}{ds^2} + \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\sigma}{ds}\frac{dx^\nu}{ds}\right)\delta x^\mu\,ds$$

$$\Rightarrow g_{\mu\nu}\frac{d^2x^\nu}{ds^2} = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\sigma} - 2\frac{\partial g_{\sigma\nu}}{\partial x^\mu}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

指标轮换，并利用 $g_{\mu\nu}$ 对称

$$g_{\mu\rho}\frac{d^2x^\rho}{ds^2}=\frac{1}{2}\left(\frac{\partial g_{\rho\sigma}}{\partial x^\mu}-\frac{\partial g_{\mu\sigma}}{\partial x^\rho}-\frac{\partial g_{\mu\rho}}{\partial x^\sigma}\right)\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}$$

即得到测地线方程

对狭义相对论情形，$g_{\mu\nu}=\eta_{\mu\nu}$ 各分量为 ±1

$RHS=0 \Rightarrow \frac{d^2x^\rho}{ds^2}=0$ 给出世界线是直线

$$\begin{cases} t=c_0\tau+t_0 \\ x_i=c_i\tau+x_{i0} \end{cases}$$

由 $d\tau=\sqrt{1-\frac{v^2}{c^2}}\,dt$ 以及间隔不变，

$$\sum_{i=1}^{3}c_i^2-c_0^2=-1$$

得到 $c_0=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$　　$c_i=\frac{v_i/c_i}{\sqrt{1-v^2/c^2}}$