2.

(a) $S'=\int_{t_1}^{t_2}L'dt=\int_{t_1}^{t_2}L\,dt+\int_{t_1}^{t_2}\frac{df}{dt}dt=S+f(q,t)\Big|_{q_1,t_1}^{q_2,t_2}$

$\delta S'=\delta S+\delta f\Big|_{q_1,t_1}^{q_2,t_2}$，由于变分要求 $\delta q_1=\delta q_2=0$，进而 $\delta f=0$

故而 $\delta S=0\Rightarrow\delta S'=0$. 即运动方程不改变.

(b) $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot q}\right)_q-\left(\frac{\partial L'}{\partial q}\right)_{\dot q}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)_q-\left(\frac{\partial L}{\partial q}\right)_{\dot q}+\frac{d}{dt}\left[\frac{\partial}{\partial \dot q}\left(\frac{df}{dt}\right)\right]_q-\left[\frac{\partial}{\partial q}\left(\frac{df}{dt}\right)\right]_{\dot q}$

$=0+\frac{d}{dt}\left[\frac{\partial}{\partial \dot q}\left(\frac{\partial f}{\partial q}\dot q+\frac{\partial f}{\partial t}\right)\right]_q-\left[\frac{\partial}{\partial q}\left(\frac{\partial f}{\partial q}\dot q+\frac{\partial f}{\partial t}\right)\right]_{\dot q}$

$=\frac{d}{dt}\left(\frac{\partial f}{\partial q}\right)_t-\frac{\partial^2 f}{\partial q^2}\dot q-\frac{\partial^2 f}{\partial q\partial t}=\frac{\partial^2 f}{\partial q^2}\dot q+\frac{\partial^2 f}{\partial q\partial t}-\frac{\partial^2 f}{\partial q^2}\dot q-\frac{\partial^2 f}{\partial q\partial t}=0.$

注意第三个等号中有：$\left[\frac{\partial}{\partial \dot q}\left(\frac{\partial f}{\partial q}\dot q\right)\right]_q=\frac{\partial f}{\partial q}.$

3.

Lorentz 变换（除去平移）满足：$\eta_{\mu\nu}=\Lambda^{\rho}{}_{\mu}\Lambda^{\phi}{}_{\nu}\eta_{\rho\phi}$.

则 $d\tau^2=-\eta_{\mu\nu}dx^\mu dx^\nu$ $\quad d\tau'^2=-\eta_{\mu\nu}dx'^\mu dx'^\nu=-\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\phi}dx^\rho dx^\phi=-\eta_{\rho\phi}dx^\rho dx^\phi=d\tau^2$

可见 $d\tau$ 是 Lorentz 变换下的不变量.

进而：$L'=\frac{1}{2}m\eta_{\mu\nu}\frac{dx'^\mu}{d\tau'}\frac{dx'^\nu}{d\tau'}=\frac{1}{2}m\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\phi}\frac{dx^\rho}{d\tau'}\frac{dx^\phi}{d\tau'}=\frac{1}{2}m\eta_{\rho\phi}\frac{dx^\rho}{d\tau'}\frac{dx^\phi}{d\tau'}$

$\qquad=\frac{1}{2}m\eta_{\rho\phi}\frac{dx^\rho}{d\tau}\frac{dx^\phi}{d\tau}=L.$

可见 L 满足 Lorentz 不变性，是一个合理的候选者.

下面只需证明 L 给出了恰当的运动方程：

$\frac{\partial L}{\partial(dx^\mu/d\tau)}=m\eta_{\mu\nu}\frac{dx^\nu}{d\tau}\Rightarrow\frac{d}{d\tau}\left(\frac{\partial L}{\partial\cdots}\right)=m\eta_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right)-\frac{\partial \mathcal{L}}{\partial x^\mu}=0 \Rightarrow m\eta_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}=0 \Rightarrow \frac{d^2x^\nu}{d\tau^2}=0$$

这个运动方程是正确的，说明我们的拉氏量是可行的.

进一步的说明：

(1) $\frac{d^2x^\mu}{d\tau^2}=0 \Rightarrow \frac{d^2x^i}{d\tau^2}=\frac{d}{d\tau}\left(\frac{d}{d\tau}x^i\right)=\frac{d}{d\tau}\left(\frac{dt}{d\tau}\dot{x}^i\right)=\frac{dt}{d\tau}\left[\frac{d}{dt}\left(\frac{dt}{d\tau}\right)\right]\dot{x}^i+\left(\frac{dt}{d\tau}\right)^2\ddot{x}^i$

而又 $\frac{d^2x^0}{d\tau^2}=0 \Rightarrow \frac{dx^0}{d\tau}=\frac{dt}{d\tau}=\text{const} \Rightarrow \left(\frac{dt}{d\tau}\right)^2\ddot{x}^i=0 \Rightarrow \ddot{x}^i=0$、

可见在3D下我们的结论也是正确的.

(2) 拉氏量的定义不是某个具体的函数，而是能导出正确的运动方程的函数. 事实上以下三个拉氏量都可以作为自由粒子拉氏量：

$$\mathcal{L}=\frac{1}{2}\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \qquad \mathcal{L}=-m\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

$$\mathcal{L}=\frac{1}{e}\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}-m^2e$$ e不是自然常数而是一个辅助变量.

特别地，很多同学试着仿照胡道的做法用坐标变换来推这个作用量. 这里也给一个参考做法.

(来自周双勇老师2023春广义相对论讲义).

Velocity addition law

$$\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \gamma\mathbf{v} + \frac{\gamma-1}{v^2}(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\mathbf{v})\mathbf{v} \tag{1.13}$$

$$\frac{\mathrm{d}t'}{\mathrm{d}t}\mathbf{u}' = \mathbf{u} + \gamma\mathbf{v} + \frac{\gamma-1}{v^2}(\mathbf{u}\cdot\mathbf{v})\mathbf{v} \tag{1.14}$$

$\mathrm{d}t'/\mathrm{d}t = \gamma\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)$

Poincare transformations connect inertial frames in special relativity

`Poincare transformations = inhomogeneous Lorentz transformations`

- 1 temporal translation: $t' = t + c_t$
- 3 spatial translations: $\mathbf{r}' = \mathbf{r} + \mathbf{c}$
- 3 rotations: $\mathbf{r}' = O\mathbf{r}$, O being SO(3) matrix,
- 3 Lorentz boosts: $\mathbf{r}' = \mathbf{r} + \gamma\mathbf{v}t + \frac{\gamma-1}{v^2}(\mathbf{r}\cdot\mathbf{v})\mathbf{v}$, $t' = \gamma\left(t + \frac{\mathbf{r}\cdot\mathbf{v}}{c^2}\right)$,

Lorentz transformations = rotations + Lorentz boosts

Lorentz boosts are called Lorentz transformations in narrow sense

Lorentz transformations are linear transformations

As we will see more clearly later, Lorentz boosts being the right symmetry implies that: In SR, we have isotropy of (4 dimensional) spacetime, instead of isotropy of space + Galilean boosts as in Newtonian mechanics.

of course we still have homogeneity of spacetime, same as in Newtonian mechan.

Einstein's relativity principle

For an isolated system in inertial frames, its action is invariant under Poincare transformations.

aka Poincare invariance or often Lorentz invariance

Lagrangian in inertial frames

One free massive particle:

we still have homogeneity of space and time and isotropy of space

We can arrive at $L = L(\dot{\mathbf{r}}^2) = L(u^2)$

Now, consider a small Lorentz boost: $\mathbf{u} \xrightarrow{\mathbf{v}=\boldsymbol{\epsilon}\to 0} \mathbf{u}'$

`frame` K `has velocity` $\mathbf{v}$ `relative to frame` K'

Note that $\mathrm{d}t' \neq \mathrm{d}t$, so we should look at $\quad \mathbf{u}' = \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t'}, \quad \mathbf{u} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$

$$\delta S \equiv \int_{t_I}^{t_F} \left[L\left(\mathbf{u}'^2\right) \frac{\mathrm{d}t'}{\mathrm{d}t} \mathrm{d}t - L\left(\mathbf{u}^2\right) \mathrm{d}t \right] = \int \left[L\left(\mathbf{u}'^2\right) \mathrm{d}t' - L\left(\mathbf{u}^2\right) \mathrm{d}t \right], \tag{1.15}$$

$$= \int \delta\left(L\left(u^2\right) \mathrm{d}t\right) = \int \delta\left(L\left(u^2\right)\right) \mathrm{d}t + \int L\left(u^2\right) \delta\left(\mathrm{d}t\right) \tag{1.16}$$

`alternatively, consider` $S = \int_{t_I}^{t_F} \mathrm{d}t L = \int_{\tau_I}^{\tau_F} \mathrm{d}\tau \frac{\mathrm{d}t}{\mathrm{d}\tau} L$ `with proper time` τ

`for action to be invariant, must be that` $\delta(\frac{\mathrm{d}t}{\mathrm{d}\tau} L) = \mathrm{d}(...)/\mathrm{d}\tau \implies \delta(\mathrm{d}t L) = \mathrm{d}(...)$

`btw, we typically consider` $t_I \to -\infty,\ t_F \to \infty$`,` $x_I^i \to -\infty,\ x_F^i \to \infty$

Since $\gamma = 1 + \frac{\epsilon^2}{2c^2} + O(\epsilon^4)$, keeping only $O(\epsilon)$ terms,

`Lorentz boost formula` $\frac{\mathrm{d}t'}{\mathrm{d}t}\mathbf{u}' = \mathbf{u} + \gamma\boldsymbol{\epsilon} + \frac{\gamma-1}{\epsilon^2}(\mathbf{u}\cdot\boldsymbol{\epsilon})\boldsymbol{\epsilon}, \quad \mathrm{d}t' = \gamma\left(1 + \frac{\mathbf{u}\cdot\boldsymbol{\epsilon}}{c^2}\right)\mathrm{d}t$

$$\delta\mathbf{u} = \mathbf{u}' - \mathbf{u} = (\mathbf{u} + \boldsymbol{\epsilon})\left(1 - \frac{\mathbf{u}\cdot\boldsymbol{\epsilon}}{c^2}\right) - \mathbf{u} = \boldsymbol{\epsilon} - \frac{\mathbf{u}\cdot\boldsymbol{\epsilon}}{c^2}\mathbf{u} \tag{1.17}$$

$\implies$

$$\delta\left(L\left(u^2\right)\right) = \frac{\partial L}{\partial(u^2)} 2\mathbf{u}\cdot\delta\mathbf{u} = 2\frac{\partial L}{\partial(u^2)}\left(1 - \frac{u^2}{c^2}\right)\mathbf{u}\cdot\boldsymbol{\epsilon} \tag{1.18}$$

$$\delta(\mathrm{d}t) = \mathrm{d}t' - \mathrm{d}t = \frac{\mathbf{u}\cdot\boldsymbol{\epsilon}}{c^2}\mathrm{d}t \tag{1.19}$$

therefore

$$\delta S = \int \mathrm{d}t \left(2\frac{\partial L}{\partial(u^2)}\left(1 - \frac{u^2}{c^2}\right) + \frac{L(u^2)}{c^2}\right)\mathbf{u}\cdot\boldsymbol{\epsilon} \tag{1.20}$$

For the integrand to be a total derivative, we need

$$2\frac{\partial L}{\partial(u^2)}\left(1 - \frac{u^2}{c^2}\right) + \frac{L(u^2)}{c^2} = A = const \tag{1.21}$$

`here` ∂ `is actually` d `in the usual notation`

$\implies$

$$L(u^2) = Ac^2 + B\sqrt{1 - \frac{u^2}{c^2}} \tag{1.22}$$

B `is a constant`

`If we choose` $A = 0$`, then` $L\mathrm{d}t$ `is invariant under Lorentz boosts`

So for a free massive relativistic particle, we can choose action

$$S = -mc^2 \int \mathrm{d}t \sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}} = \int \mathrm{d}t \left(-mc^2 + \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{m\dot{\mathbf{r}}^4}{8c^2} + ...\right) \tag{1.23}$$

EoM:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{\mathbf{r}}}{\sqrt{1-\frac{u^2}{c^2}}}\right)=0 \implies \mathbf{r}(t)=\mathbf{r}_0+\mathbf{v}_0 t \implies \text{existence of inertial frame} \tag{1.24}$$

Exercise: `derive the Hamiltonian/energy of massive relativistic particle`

`Can` $L=\sqrt{1-\frac{\dot{\mathbf{r}}^2}{c^2}}-V(\mathbf{r},t)$ `be Poincare invariant or Lorentz invariant?`

Remark:

This is for massive particles. As we see later, in relativity there are massless particles.

`mass measures inertia; no massless particles in Newtonian physics`

`in Newtonian physics, massless particle would have infinite acceleration`

`in SR, massless particles such as photons play a central role`

Lorentz force (interacting with Lorentz invariant fields):

$$L=-mc^2\sqrt{1-\frac{\dot{\mathbf{r}}^2}{c^2}}-q\phi+q\dot{\mathbf{r}}\cdot\mathbf{A} \tag{1.25}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}(m\gamma\dot{\mathbf{r}})=q\mathbf{E}+q\dot{\mathbf{r}}\times\mathbf{B} \tag{1.26}$$

`this is Lorentz invariant, as we will see later`

Curvilinear frames

`mechanical laws can be very complicated in an arbitrary frame`

`use simplest frames to deduce physical laws, then transform to other frames`

In a curvilinear coordinate system, which is not an inertial frame, the Lagrangian for a free relativistic particle is

$$L=\sqrt{1-\sum_{i,j}g_{ij}(y)\dot{y}_i\dot{y}_j} \tag{1.27}$$

We get a nontrivial metric $g_{ij}(y)$, which will be similar to curved space.
For example, in spherical coordinates $L=\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-r^2\sin^2\theta\dot{\phi}^2}$

`We usually avoid these frames for the sake of simplicity and not punishing ourselves,`

`but in curved spaces we are forced to work with these frames.`

`We can only locally choose inertial frames, which underlines the equivalence principle.`

4. $\mathcal{L} = \frac{1}{2} m \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + q A_\mu \frac{dx^\mu}{d\tau}$

下面我们大概看一下怎么搞，在低速时，有：

$\mathcal{L} = \frac{1}{2} m v^2 - q\phi + q \vec{v} \cdot \vec{A}$

注意到：$\phi = A^0 \quad -1 \simeq -\frac{dt}{d\tau} = \frac{dx_0}{d\tau}$（注意是 x_0 而非 x^0） $\quad v^i = \frac{dx^i}{dt}$

则相互作用项可以被写作：$q \frac{dx^\mu}{dt} A^\mu = q A_\mu \frac{dx^\mu}{dt}$

而自由项应被直接换为SR的自由项。当然，我们知道一个选择是：

$\mathcal{L} = -m\sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} + q A_\mu \frac{dx^\mu}{dt}$

我们知道，只需让 S 不变即可，这告诉我们 $\frac{d}{dt}$ 可以被换为 $\frac{d}{d\tau}$，即：

$S = \int_{t_1}^{t_2} \left[-m\sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} + q A_\mu \frac{dx^\mu}{dt} \right] dt = \int_{\tau_1}^{\tau_2} \left[-m\sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} + q A_\mu \frac{dx^\mu}{d\tau} \right] d\tau$

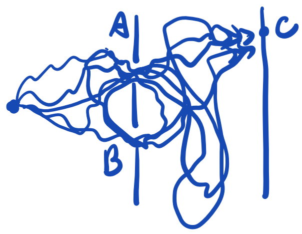
如果这样的 $\mathcal{L}$ 可以导出正确的运动方程，那以下的 $\mathcal{L}$ 也一定可以：

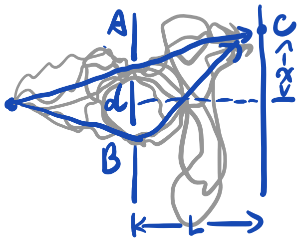
$\mathcal{L} = \frac{1}{2} m \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + q A_\mu \frac{dx^\mu}{d\tau}$

因为他们的自由项导出的运动方程是一样的。

1. 取屏上一点C，电子通过一切可能路径到达C。如下图所示。在所有路径中，只有两条经典路径附近的概率幅对结果贡献了主要的主要的概率。对于这两个概率幅来说：

$$V \propto e^{iEt_1} + e^{iEt_2}$$





$$p \propto VV^* = 2 + e^{iE(t_1-t_2)} + e^{-iE(t_1-t_2)}$$
$$= 2 + 2\cos[E(t_1-t_2)]$$

$$vt_1 = \sqrt{L^2+(x-\frac{d}{2})^2} \simeq L\left[1+\frac{1}{2}\left(\frac{x-\frac{d}{2}}{L}\right)^2\right]$$
$$vt_2 = \sqrt{L^2+(x+\frac{d}{2})^2} \simeq L\left[1+\frac{1}{2}\left(\frac{x+\frac{d}{2}}{L}\right)^2\right]$$

$\Rightarrow t_1 - t_2 \propto x$. 从而会随C的位置变动而形成干涉条纹.

5. 我认为以下几种说法都有道理.

(1) 我们需要L，是因为没有运动方程(以后都简称EoM)而我们要求解它. 此时自然没有$q(t)$，不存在$\dot{q}=\frac{dq}{dt}$；若已知了$q(t)$，我们也不需要L了.

(2) L是一个态函数，对应于某一时刻的状态. 在这一时刻q和$\dot{q}$(也即速度)是两个状态量，不存在说什么导数的关系.

(3) 运动方程必是二阶微分方程，两个初值条件缺一不可. 对我们的测量来说，必应知道的就是q和$\dot{q}$