《理论力学 A》期末考试参考答案（20200108）

一、第一题

1.

$$
\begin{aligned}
L &= \frac{1}{2}mv^2 - e(\phi - \vec{v}\cdot\vec{A}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) + e\vec{v}\cdot\vec{A} \\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) + eb(1-\cos\theta)\dot{\varphi} \qquad (1)
\end{aligned}
$$

运动积分:

(1) φ 为循环坐标，所以与其共轭的广义动量为运动积分：

$$
p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \mathrm{m}r^2\sin^2\theta\dot{\varphi} + eb(1-\cos\theta) = \text{ const.} \qquad (2)
$$

(2) L 不显含时间，所以广义能量（哈密顿量）为运动积分：

$$
\mathrm{H} = p_r\dot{r} + p_\theta\dot{\theta} + p_\varphi\dot{\varphi} - L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) = \text{ const} \qquad (3)
$$

2. 直角坐标系与球坐标系的关系有 $x=\tan\varphi$，所以有 $\dot{\varphi} = \frac{x\dot{y}-\dot{x}y}{x^2+y^2}$。因此直角坐标系下拉格朗日函数为:

$$
L = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + eb\frac{r-z}{r}\frac{x\dot{y}-\dot{x}y}{x^2+y^2} = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + \frac{eb}{r}\frac{x\dot{y}-\dot{x}y}{r+z} \qquad (4)
$$

其中 $r=\sqrt{x^2+y^2+z^2}$。可得

$$
p_x = m\dot{x} - \frac{eb}{r(r+z)}y, p_y = m\dot{y} + \frac{eb}{r(r+z)}x, p_z = m\dot{z} \qquad (5)
$$

不难获得哈密顿函数

$$
H = \frac{1}{2m}\left[\left(p_x + \frac{eb}{r(r+z)}y\right)^2 + \left(p_y - \frac{eb}{r(r+z)}x\right)^2 + p_z^2\right] \qquad (6)
$$

所求正则方程组为

$$
\begin{aligned}
\dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad (7) \\
\dot{p}_z &= -\frac{\partial H}{\partial z} = -\frac{eb}{m}\left[\left(p_x + \frac{eb}{r(r+z)}y\right)y - \left(p_y - \frac{eb}{r(r+z)}x\right)x\right]\frac{\partial}{\partial z}\frac{1}{r(r+z)} \\
&= \frac{eb}{m}\left[yp_x - xp_y + \frac{eb(r-z)}{r}\right]\frac{z(2r+z)+r^2}{r^3(r+z)^2} \\
&= \frac{eb}{mr^3}\left[yp_x - xp_y + \frac{eb(r-z)}{r}\right] \qquad (8)
\end{aligned}
$$

二、第二题

(a) 变换关系

$$
\begin{cases} x = x_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2 \\ p = p_0 + mgt \end{cases} \qquad (9)
$$

$$
\begin{cases} x_0 = x - \frac{p}{m}t + \frac{1}{2}gt^2 \\ p_0 = p - mgt \end{cases} \qquad (10)
$$

判断可积性

$$\begin{aligned}\delta F &= p\delta x - p_0\delta x_0 = p\delta x - (p-mgt)\delta\left(x-\frac{p}{m}t+\frac{1}{2}gt^2\right)\\ &= mgt\delta x + \left(\frac{p}{m}t-gt^2\right)\delta p\end{aligned} \tag{11}$$

即:

$$F = mgxt + \frac{p^2}{2m}t - pgt^2 \tag{12}$$

所以是正则变换。或者:

$$\frac{\partial(x_0,p_0)}{\partial(x,p)} = 1 \tag{13}$$

因此是正则变换。

(b) 第三类生成函数 $F_3(p,x_0,t)$

$$\frac{\partial x_0}{\partial x} = 1 \neq 0 \tag{14}$$

所以 $F_3(p,x_0,t)$ 存在，生成函数为

$$\left.\begin{aligned} F_3(t,p,x_0) = F - xp = mgxt & +\tfrac{p^2}{2m}t - pgt^2 - xp\\ x & = x_0 + \tfrac{p}{m}t - \tfrac{1}{2}gt^2 \end{aligned}\right\}\Rightarrow \tag{15}$$

$$F_3(t,p,x_0) = -\frac{p^2}{2m}t + \frac{1}{2}gt^2p - px_0 + mgtx_0 - \frac{1}{2}mg^2t^3 \tag{16}$$

旧的哈密顿函数为:

$$H = \frac{p^2}{2m} - mgx \tag{17}$$

新的哈密顿函数为:

$$\widetilde{H} = H + \frac{\partial F_3}{\partial t} = -mg^2t^2 \tag{18}$$

第四类生成函数 $F_4(p,p_0,t)$

$$\frac{\partial p_0}{\partial x} = 0 \tag{19}$$

所以第四类生成函数 $F_4(p,p_0,t)$ 不存在。

三、第三题

(a) 系统的拉格朗日量为:

$$L = \frac{I_1}{2}\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right)^2 + \frac{I_3}{2}\left(\dot\psi + \dot\varphi\cos\theta\right)^2 - mgl\cos\theta \tag{20}$$

广义动量为:

$$\begin{aligned} P_\theta &= I_1\dot\theta\\ P_\varphi &= (I_1\sin^2\theta + I_3\cos^2\theta)\dot\varphi + I_3\cos\theta\dot\psi\\ P_\psi &= I_3(\dot\psi + \cos\theta\dot\varphi)\end{aligned} \tag{21}$$

系统存在三个运动积分: E, P_φ, P_ψ。

通过勒让德变换，得到系统的哈密顿量:

$$H = \frac{1}{2I_1}\left[\frac{(P_\varphi - P_\psi\cos\theta)^2}{\sin^2\theta} + P_\theta^2\right] + \frac{P_\psi^2}{2I_3} + mgl\cos\theta \tag{22}$$

令：

$$S=-Et+P_{\varphi}\varphi+P_{\psi}\psi+W_{\theta}(\theta) \tag{23}$$

代入哈密顿-雅可比方程，得：

$$E=\frac{1}{2I_1}\left[\frac{\left(P_{\varphi}-P_{\psi}\cos\theta\right)^2}{\sin^2\theta}+\left(\frac{\partial W_{\theta}}{\partial\theta}\right)^2\right]+\frac{P_{\psi}^2}{2I_3}+mgl\cos\theta \tag{24}$$

求得：

$$\frac{dW_{\theta}}{d\theta}=\pm\sqrt{2I_1\left(E-\frac{P_{\psi}^2}{2I_3}-mgl\cos\theta\right)-\frac{\left(P_{\varphi}-P_{\psi}\cos\theta\right)^2}{\sin^2\theta}}\equiv\pm\sqrt{\Delta(\theta;E,P_{\varphi},P_{\psi})} \tag{25}$$

最终得到：

$$S=-Et+P_{\varphi}\varphi+P_{\psi}\psi\pm\int^{\theta}\sqrt{\Delta(\theta';E,P_{\varphi},P_{\psi})}d\theta' \tag{26}$$

由 $Q_{\alpha}=\frac{\partial S}{\partial P_{\alpha}}=const.$，$P_{\alpha}$ 分别取 (E,P_{φ},P_{ψ})，得到如下的运动方程：

$$\begin{aligned}
0&=\frac{\partial S}{\partial E}=-t\pm\int^{\theta}\frac{I_1}{\sqrt{\Delta(\theta';E,P_{\varphi},P_{\psi})}}d\theta'\\
0&=\frac{\partial S}{\partial P_{\varphi}}=-t\pm\int^{\theta}\frac{-\left(P_{\varphi}-P_{\psi}\cos\theta'\right)/\sin^2\theta'}{\sqrt{\Delta(\theta';E,P_{\varphi},P_{\psi})}}d\theta'\\
0&=\frac{\partial S}{\partial P_{\psi}}=-t\pm\int^{\theta}\frac{-\frac{I_1}{I_3}P_{\psi}+\frac{\left(P_{\varphi}-P_{\psi}\cos\theta'\right)}{\sin^2\theta'}\cos\theta'}{\sqrt{\Delta(\theta';E,P_{\varphi},P_{\psi})}}d\theta'
\end{aligned} \tag{27}$$

(b)

四、第四题

方法一

(a)

$$\begin{aligned}
&[x_1,L_1]=[x_1,x_2p_3-x_3p_2]=0\\
&[x_1,L_2]=[x_1,x_3p_1-x_1p_3]=[x_1,x_3p_1]=x_3\\
&[x_1,L_3]=[x_1,x_1p_2-x_2p_1]=-[x_1,x_2p_1]=-x_2
\end{aligned} \tag{28}$$

同理可证：

$$\begin{aligned}
&[x_2,L_1]=-x_3,\quad [x_2,L_2]=0,\quad [x_2,L_3]=x_1\\
&[x_3,L_1]=x_2,\quad [x_3,L_2]=-x_1,\quad [x_3,L_3]=0
\end{aligned} \tag{29}$$

即：

$$[x_i,L_j]=\varepsilon_{ijk}x_k \tag{30}$$

$$\begin{aligned}
&[p_1,L_1]=[p_1,x_2p_3-x_3p_2]=0\\
&[p_1,L_2]=[p_1,x_3p_1-x_1p_3]=-p_3\,[p_1,x_1]=p_3\\
&[p_1,L_3]=[p_1,x_1p_2-x_2p_1]=p_2\,[p_1,x_1]=-p_2
\end{aligned} \tag{31}$$

同理可证：

$$\begin{aligned}
&[p_2,L_1]=-p_3,\quad [p_2,L_2]=0,\quad [p_2,L_3]=p_1\\
&[p_3,L_1]=p_2,\quad [p_3,L_2]=-p_1,\quad [p_3,L_3]=0
\end{aligned} \tag{32}$$

即:

$$[p_i, L_j] = \varepsilon_{ijk} p_k \tag{33}$$

$$\begin{aligned}[L_1, L_1] &= [x_2p_3 - x_3p_2, x_2p_3 - x_3p_2] \\ &= [x_2p_3, -x_3p_2] - [x_3p_2, x_2p_3] = 0\end{aligned} \tag{34}$$

$$\begin{aligned}[L_1, L_2] &= [x_2p_3 - x_3p_2, x_3p_1 - x_1p_3] \\ &= [x_2p_3, x_3p_1] + [x_2p_3, -x_1p_3] + [-x_3p_2, x_3p_1] + [x_3p_2, x_1p_3] \\ &= x_2p_1\,[p_3, x_3] + 0 + 0 + x_1p_2\,[x_3, p_3] \\ &= -x_2p_1 + x_1p_2 = L_3\end{aligned} \tag{35}$$

同理可证:

$$[L_1, L_3] = -L_2, \quad [L_2, L_3] = L_1 \tag{36}$$

即:

$$[L_i, L_j] = \varepsilon_{ijk} L_k \tag{37}$$

(b)

$$\begin{aligned}[j_i, j_j] &= \frac{1}{\sqrt{GMa}}\frac{1}{\sqrt{GMa}}[L_i, L_j] \\ &= \frac{1}{\sqrt{GMa}} \cdot \frac{1}{\sqrt{GMa}}\varepsilon_{ijk}L_k \\ &= \frac{1}{\sqrt{GMa}}\varepsilon_{ijk}j_k\end{aligned} \tag{38}$$

$$[j_1, e_1] = \frac{1}{\sqrt{GMa}}\left[L_1, \frac{1}{GM}(p_2L_3 - p_3L_2) - \frac{x_1}{r}\right] \tag{39}$$

先计算:

$$\begin{aligned}[L_1, p_2L_3 - p_3L_2] &= [L_1, p_2L_3] - [L_1, p_3L_2] \\ &= [L_1, p_2]\,L_3 + p_2\,[L_1, L_3] - [L_1, p_3]\,L_2 - p_3\,[L_1, L_2] \\ &= p_3L_3 - p_2L_2 + p_2L_2 - p_3L_3 = 0\end{aligned} \tag{40}$$

$$\left[L_1, \frac{x_1}{r}\right] = \frac{1}{r}[L_1, x_1] + x_1\left[L_1, \frac{1}{r}\right] = x_1\left[L_1, \frac{1}{r}\right] \tag{41}$$

$$\begin{aligned}\left[L_1, \frac{1}{r}\right] &= \frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)[L_1, x_1] + \frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)[L_1, x_2] + \frac{\partial}{\partial x_3}\left(\frac{1}{r}\right)[L_1, x_3] \\ &= 0 + \left(-\frac{1}{r^3}\right)x_2x_3 + \left(-\frac{1}{r^3}\right)x_3\,(-x_2) = 0\end{aligned} \tag{42}$$

同理可证:

$$\left[L_2, \frac{1}{r}\right] = \left[L_3, \frac{1}{r}\right] = 0 \tag{43}$$

因此:

$$[j_1, e_1] = 0 \tag{44}$$

$$[j_1, e_2] = \frac{1}{\sqrt{GMa}}\left[L_1, \frac{1}{GM}(p_3L_1 - p_1L_3) - \frac{x_2}{r}\right] \tag{45}$$

先计算:

$$[L_1, p_3L_1 - p_1L_3] = [L_1, p_3]\,L_1 - p_1\,[L_1, L_3] = -p_2L_1 + p_1L_2 = (\vec{p} \times \vec{L})_3 \tag{46}$$

$$\left[L_1, -\frac{x_2}{r}\right] = -x_2\left[L_1, \frac{1}{r}\right] + \frac{1}{r}\left[L_1, -x_2\right] = -\frac{x_3}{r} \tag{47}$$

即:

$$[j_1, e_2] = e_3 \tag{48}$$

同理可证:

$$[j_1, e_3] = -e_2, \quad [j_2, e_3] = e_1 \tag{49}$$

即:

$$[j_i, e_j] = \varepsilon_{ijk}e_k \tag{50}$$

$$[e_1, e_2] = \left[\frac{1}{GM}\left(p_2L_3 - p_3L_2\right) - \frac{x_1}{r}, \frac{1}{GM}\left(p_3L_1 - p_1L_3\right) - \frac{x_2}{r}\right] \tag{51}$$

先计算:

$$\begin{aligned}
& [p_2L_3 - p_3L_2, p_3L_1 - p_1L_3] \\
= \; & [p_2L_3, p_3L_1] - [p_2L_3, p_1L_3] - [p_3L_2, p_3L_1] + [p_3L_2, p_1L_3] \\
= \; & p_2\left[L_3, p_3L_1\right] + \left[p_2, p_3L_1\right]L_3 - p_2\left[L_3, p_1L_3\right] - \left[p_2, p_1L_3\right]L_3 \\
& -p_3\left[L_2, p_3L_1\right] - \left[p_3, p_3L_1\right]L_2 + p_3\left[L_2, p_1L_3\right] + \left[p_3, p_1L_3\right]L_2 \\
= \; & p_2p_3L_2 + \left(-p_3\right)p_3L_3 - p_2^2L_3 - p_1^2L_3 \\
& -p_3\left[L_2, p_3\right]L_1 - p_3^2\left[L_2, L_1\right] - p_2p_3L_2 + p_3\left[L_2, p_1\right]L_3 + p_3p_1\left[L_2, L3\right] \\
= \; & p_2p_3L_2 - p_3^2L_3 - p_2^2L_3 - p_1^2L_3 \\
& -p_1p_3L_1 + p_3^2L_3 - p_2p_3L_2 + p_3\left(-p_3\right)L_3 + p_1p_3L_1 \\
= \; & -p^2L_3
\end{aligned} \tag{52}$$

继续计算:

$$\begin{aligned}
& \left[p_2L_3 - p_3L_2, \frac{x_2}{r}\right] \\
= \; & \left[p_2L_3 - p_3L_2, x_2\right]\frac{1}{r} + \left[p_2L_3 - p_3L_2, \frac{1}{r}\right]x_2 \\
= \; & \left\{-L_3 - p_2x_1\right\}\frac{1}{r} + \left\{L_3\left[p_2, \frac{1}{r}\right] - L_2\left[p_3, \frac{1}{r}\right]\right\}x_2 \\
= \; & \left\{-L_3 - x_1p_2\right\}\frac{1}{r} + \frac{1}{r^3}\left\{x_2L_3 - x_3L_2\right\}x_2
\end{aligned} \tag{53}$$

$$\begin{aligned}
& \left[\frac{x_1}{r}, p_3L_1 - p_1L_3\right] \\
= \; & \frac{1}{r}\left[x_1, p_3L_1 - p_1L_3\right] + x_1\left[\frac{1}{r}, p_3L_1 - p_1L_3\right] \\
= \; & \frac{1}{r}\left\{-L_3 + x_2p_1\right\} + x_1\left\{L_1\left[\frac{1}{r}, p_3\right] - L_3\left[\frac{1}{r_1}, p_1\right]\right\} \\
= \; & \frac{1}{r}\left\{-L_3 + x_2p_1\right\} + x_1\frac{1}{r^3}\left\{-L_1x_3 + L_3x_1\right\}
\end{aligned} \tag{54}$$

两式相加:

$$\begin{aligned} & \frac{1}{r}\{-2L_3+p_1x_2-p_2x_1\}+\frac{1}{r^3}\left\{(x_1^2+x_2^2)L_3-x_1x_3L_1-x_2x_3L_2\right\} \\ = & \frac{1}{r}\{-3L_3\}+\frac{1}{r^3}\left\{(x_1^2+x_2^2)L_3-x_1x_3L_1-x_2x_3L_2\right\} \\ = & -\frac{2}{r}L_3-\frac{x_3}{r^3}\{x_3L_3+x_1L_1+x_2L_2\} \\ = & -\frac{2}{r}L_3-\frac{x_3}{r^3}\vec{r}\cdot\vec{L} \end{aligned} \tag{55}$$

因此,

$$\begin{aligned} [e_1,e_2] &= \frac{1}{(GM)^2}\left(-p^2L_3\right)+\frac{1}{GM}\left\{\frac{2}{r}L_3+\frac{x_3}{r^3}\vec{r}\cdot\vec{L}\right\} \\ &= \frac{L_3}{(GM)^2}\left(-p^2+\frac{2GM}{r}\right)+\frac{1}{GM}\cdot\frac{x_3}{r^3}(\vec{r}\cdot\vec{L}) \\ &= \frac{L_3}{(GM)^2}(-2E)+\frac{1}{GM}\frac{x_3}{r^3}(\vec{r}\cdot\vec{L}) \end{aligned} \tag{56}$$

利用:

$$E=-\frac{GM}{2a}, \quad \vec{r}\cdot\vec{L}=0 \tag{57}$$

可得:

$$[e_1,e_2]=\frac{L_3}{GMa}=\frac{1}{\sqrt{GMa}}j_3 \tag{58}$$

同理可证:

$$[e_1,e_3]=-\frac{1}{\sqrt{GMa}}j_2, \quad [e_2,e_3]=\frac{1}{\sqrt{GMa}}j_1 \tag{59}$$

即:

$$[e_i,e_j]=\frac{1}{\sqrt{GMa}}\varepsilon_{ijk}j_k \tag{60}$$

方法二

(a)

$$\begin{aligned} [x_i,L_j] &= [x_i,\varepsilon_{jlm}x_lp_m] \\ &= \frac{\partial}{\partial p_i}[\varepsilon_{jlm}x_lp_m] \\ &= \varepsilon_{jlm}x_l\delta_{im} \\ &= \varepsilon_{jli}x_l=\varepsilon_{ijl}x_l \end{aligned} \tag{61}$$

$$\begin{aligned} [p_i,L_i] &= -\frac{\partial}{\partial x_i}L_j=-\frac{\partial}{\partial x_i}\varepsilon_{jlm}x_lp_m \\ &= -\varepsilon_{jlm}p_m\delta_{il} \\ &= -\varepsilon_{jim}p_m=\varepsilon_{ijm}p_m \end{aligned} \tag{62}$$

$$\begin{aligned}
[L_i, L_j] &= [\varepsilon_{ilm} x_l p_m, L_j] = \varepsilon_{ilm} x_l [p_m, L_j] + \varepsilon_{ilm} [x_l, L_j] p_m \\
&= \varepsilon_{ilm}\varepsilon_{mjn} x_l p_n + \varepsilon_{ilm}\varepsilon_{ljn} x_n p_m \\
&= \varepsilon_{ilm}\varepsilon_{mjn} x_l p_n + \varepsilon_{imn}\varepsilon_{mjl} x_l p_n \\
&= (\varepsilon_{mil}\varepsilon_{mjn} + \varepsilon_{mni}\varepsilon_{mjl}) x_l p_n \\
&= (\delta_{ij}\delta_{ln} - \delta_{in}\delta_{lj} + \delta_{nj}\delta_{il} - \delta_{nl}\delta_{ij}) x_l p_n \\
&= (\delta_{il}\delta_{nj} - \delta_{in}\delta_{lj}) x_l p_n \\
&= \varepsilon_{ijk}\varepsilon_{lnk} x_l p_n \\
&= \varepsilon_{ijk} L_k
\end{aligned} \tag{63}$$

(b)

$$\begin{aligned}
[j_i, j_j] &= \frac{1}{\sqrt{GMa}} \frac{1}{\sqrt{GMa}} [L_i, L_j] \\
&= \frac{1}{\sqrt{GMa}} \cdot \frac{1}{\sqrt{GMa}} \varepsilon_{ijk} L_k \\
&= \frac{1}{\sqrt{GMa}} \varepsilon_{ijk} j_k
\end{aligned} \tag{64}$$

$$\begin{aligned}
[j_i, e_j] &= \frac{1}{\sqrt{GMa}} \left[L_i, \frac{1}{GM} \varepsilon_{jlm} p_l L_m - \frac{x_j}{r} \right] \\
&= \frac{1}{\sqrt{GMa}} \frac{1}{GM} [L_i, \varepsilon_{jlm} p_l L_m] - \frac{1}{\sqrt{Gma}} \left[L_i, \frac{x_j}{r} \right]
\end{aligned} \tag{65}$$

分别计算第一项：

$$\begin{aligned}
&[L_i, \varepsilon_{jlm} p_l L_m] = \varepsilon_{jlm} [L_i, p_l] L_m + \varepsilon_{jlm} p_l [L_i, L_m] \\
&= \varepsilon_{jlm}\varepsilon_{iln} p_n L_m + \varepsilon_{jlm}\varepsilon_{imn} p_l L_n \\
&= (\varepsilon_{jmn}\varepsilon_{iml} + \varepsilon_{jlm}\varepsilon_{imn}) p_k L_n \\
&= (\delta_{ji}\delta_{nl} - \delta_{jl}\delta_{ni} + \delta_{jn}\delta_{li} - \delta_{ji}\delta_{ln}) p_l L_n \\
&= (\delta_{jn}\delta_{li} - \delta_{jl}\delta_{ni}) p_l L_n \\
&= \varepsilon_{ijk}\varepsilon_{kln} p_l L_n = \varepsilon_{ijk} (\vec{p} \times \vec{L})_k
\end{aligned} \tag{66}$$

第二项：

$$\begin{aligned}
&\left[L_i, \tfrac{x_i}{r}\right] = \tfrac{1}{r} [L_i, x_j] + x_j \left[L_i, \tfrac{1}{r}\right] \\
&= \tfrac{1}{r} \varepsilon_{ijk} x_k + x_j \left(-\tfrac{\partial L_i}{\partial p_k}\right) \left(\tfrac{\partial}{\partial x^k} \tfrac{1}{r}\right) \\
&= \varepsilon_{ijk} \tfrac{x_k}{r} + x_j \cdot \tfrac{1}{r^3} \cdot x_k \tfrac{\partial L_i}{\partial p_k} \\
&= \varepsilon_{ijk} \tfrac{x_k}{r} + x_j \tfrac{1}{r^3} \cdot x_k [x_k, L_i] \\
&= \varepsilon_{ijk} \tfrac{x_k}{r} + x_j \cdot \tfrac{1}{r^3} \varepsilon_{ilk} x_l x_k \\
&= \varepsilon_{ijk} \tfrac{x_k}{r}
\end{aligned} \tag{67}$$

因此，

$$\begin{aligned}
[j_i, e_j] &= \frac{1}{\sqrt{GMa}} \frac{1}{GM} [L_i, \varepsilon_{jlm} p_l L_m] - \frac{1}{\sqrt{Gma}} \left[L_i, \frac{x_j}{r} \right] \\
&= \frac{1}{\sqrt{GMa}} \left(\frac{1}{GM} \varepsilon_{ijk} (\vec{p} \times \vec{L})_k - \varepsilon_{ijk} \frac{x_k}{r} \right) \\
&= \varepsilon_{ijk} e_k
\end{aligned} \tag{68}$$

五、第五题

先建立坐标系。在某时刻实验室坐标系与随体惯性系重合。选 A 为坐标原点，$\vec{AB}$ 轴为 z 轴，y 轴垂直向上，$A-xyz$ 形成右手坐标系。

根据无滑滚动条件，考虑 B 点的速度，得到自转角速度 ω 为：

$$\omega_0 l = \omega r \Rightarrow \omega = \frac{l}{r}\omega_0 \tag{69}$$

刚体的三个惯量主轴分别为：

$$I_3 = \frac{1}{2}mr^2, \quad I_1 = I_2 = \frac{1}{4}mr^2 + ml^2 \tag{70}$$

刚体的角速度的角加速度分别为：

$$\omega_1 = 0, \quad \omega_2 = \omega_0, \quad \omega_3 = -\omega = -\frac{l}{r}\omega_0 \tag{71}$$

方法一：在实验室坐标系中讨论

$$\vec{N} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left[I_2\omega_2\vec{e}_{y'} + I_3\omega_3\vec{e}_{z'}\right] = I_3\omega_3\frac{d}{dt}\vec{e}_{z'} = I_3\omega_3\omega_2\vec{e}_{x'} = -\frac{1}{2}mrl\omega_0^2\vec{e}_{x'} \tag{72}$$

由 $N_1 = -F_2 l - rF_3, \quad F_2 = N - mg, \quad F_3 = 0$，得：

$$N = \frac{1}{2}mr\omega_0^2 + mg \tag{73}$$

方法二：在随动惯性系中讨论

$$\dot{\omega}_1 = \omega_2\omega_3, \quad \dot{\omega}_2 = \dot{\omega}_3 = 0 \tag{74}$$

代入欧拉方程：

$$N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 = I_3\omega_2\omega_3 = -\frac{1}{2}mrl\omega_0^2 \tag{75}$$

$$N_2 = N_3 = 0 \tag{76}$$

由 $N_1 = -F_2 l - rF_3, \quad F_2 = N - mg, \quad F_3 = 0$，得：

$$N = \frac{1}{2}mr\omega_0^2 + mg \tag{77}$$