3.

(a) 我们直接代入悬系即可得到 L 不变，这没有什么思维上的难度.

现在我们考虑一个更有趣的办法. 取一个无穷小的洛伦兹 boost:

$\xi \to \epsilon u$, $\epsilon \ll 1$, u 为速度，这里取自然单位制. 保留到 ϵ 的一阶项.

$\cosh\xi = \cosh\epsilon u = 1 + 0\cdot\epsilon u + o(\epsilon^2)$

$\sinh\xi = \sinh\epsilon u = 0 + \epsilon u + o(\epsilon^2)$

则有: $\begin{cases} t' = t - \epsilon u x \\ x' = x - \epsilon u t \end{cases}$ ① $\Rightarrow$ $\begin{cases} \dot t' = \dot t - \epsilon u \dot x \\ \dot x' = \dot x - \epsilon u \dot t \end{cases}$ ②

则有:

$$\delta L = \frac{\partial L}{\partial \dot x}(-\epsilon u \dot t) + \frac{\partial L}{\partial \dot t}(-\epsilon u \dot x) = m\dot x(-\epsilon u \dot t) + (-m\dot t)(-\epsilon u \dot x) = m\epsilon u(-\dot x\dot t + \dot x\dot t) = 0 \quad ③$$

也即 L 在洛伦兹 boost 下不变. 根据诺特荷的公式:

(若在连续变换 $q^\mu \to q^\mu + \epsilon\Delta q^\mu$ 下, 有 $\delta L = \frac{df}{d\tau}$, 则有诺特荷 $Q = \frac{\partial L}{\partial \dot q^\mu}\Delta q^\mu - f$)

$$\tilde Q = \frac{\partial L}{\partial \dot x}(-ut) + \frac{\partial L}{\partial \dot t}(-ux) = m\dot x(-ut) - m\dot t(-ux) = mu(-\dot x t + \dot t x) \quad ④$$

(不妨取那个常数为 0)

u 具有任意性，它是我们随性选取的参数，那么实际守恒的量将是:

$$Q = m(-\dot x t + \dot t x) \quad ⑤$$

我们好奇这个 Q 等于多少呢?

$$-\dot x t + \dot t x = \frac{dt}{d\tau}\left(-\frac{dx}{dt}t + x\right) = \frac{1}{\sqrt{1-v^2}}(-vt + x). \quad ⑥$$

我们可以知道自由粒子的运动方程为: $x = x_0 + vt$ ⑦ 故而 $Q = \frac{m x_0}{\sqrt{1-v^2}}$ ⑧

说明洛伦兹 boost 对应的守恒荷是"初始位置", 和伽利略 boost 一样.

(b) 方法完全一样. 留意到 $L = L_{free} + L_{int}$, 而在洛伦兹 boost ① 和 ② 下, $\delta L_{free} = 0$.

故而只需考虑 δL_{int}.

$L_{int} = e\frac{dx^\mu}{d\tau}A_\mu$ $\quad A_\mu \equiv A_\mu(x)$ 是一个给定的场.

$$\delta L_{int} = \frac{\partial L_{int}}{\partial \dot x^\mu}\epsilon\Delta\dot x^\mu + \frac{\partial L_{int}}{\partial x^\mu}\epsilon\Delta x^\mu = eA_\mu\epsilon\Delta\dot x^\mu + e\dot x^\nu\partial_\mu A_\nu \epsilon\Delta x^\mu$$

$A_\mu\Delta\dot x^\mu = +\phi u\dot x - Au\dot t$

$$\dot x^\nu\partial_\mu A_\nu\Delta x^\mu = \dot t\left(-\frac{\partial\phi}{\partial t}\right)(-ux) + \dot t\left(-\frac{\partial\phi}{\partial x}\right)(-ut) + \dot x\left(\frac{\partial A}{\partial t}\right)(-ux) + \dot x\left(\frac{\partial A}{\partial x}\right)(-ut)$$

$$= u\dot t\left(x\frac{\partial\phi}{\partial t} + t\frac{\partial\phi}{\partial x}\right) + u\dot x\left(-x\frac{\partial A}{\partial t} - t\frac{\partial A}{\partial x}\right)$$

$$\Rightarrow \delta L_{int} = e\epsilon u\left[\dot t\left(x\frac{\partial\phi}{\partial t} + t\frac{\partial\phi}{\partial x} - A\right) + \dot x\left(\phi - x\frac{\partial A}{\partial t} - t\frac{\partial A}{\partial x}\right)\right]$$

全导数要求: $\frac{\partial}{\partial x}\left(x\frac{\partial\phi}{\partial t} + t\frac{\partial\phi}{\partial x} - A\right) = \frac{\partial}{\partial t}\left(\phi - x\frac{\partial A}{\partial t} - t\frac{\partial A}{\partial x}\right)$

$$\Rightarrow \cancel{\partial_0\phi} + x\partial_1\partial_0\phi + t\partial_1\partial_1\phi - \cancel{\partial_1 A} = \cancel{\partial_0\phi} - x\partial_0\partial_0 A - \cancel{\partial_1 A} - t\partial_1\partial_0 A \Rightarrow x(\partial_1\partial_0\phi + \partial_0\partial_0 A) + t(\partial_1\partial_1\phi + \partial_1\partial_0 A) = 0$$

$$\Rightarrow x\partial_0(\partial_1\phi + \partial_0 A) + t\partial_1(\partial_1\phi + \partial_0 A) = 0$$

令 $\frac{dt}{x}=\frac{dx}{t} \Rightarrow t\,dt - x\,dx = 0 \Rightarrow \alpha = t^2-x^2 \quad \beta = t$，原方程化为：$x\frac{\partial z}{\partial \beta}=0$

方程恒成立要求：$\frac{\partial z}{\partial \beta}=0$ 即 $z \equiv z(\alpha)$，或写：$z \equiv z(t^2-x^2)$

则最终要求：$\partial_1\phi + \partial_0 A = z(t^2-x^2)$

注意到 $E=-\partial_1\phi-\partial_0 A$，那么有 $E=z(t^2-x^2)$.

这一结果说明电场本身必须洛伦兹不变，这个解是特殊的、(体系中无磁场)

$\partial_1 E = \frac{dz}{du}(-2x) = -2x\frac{dz}{du} = \vec{\nabla}\cdot\vec{E}$ 这表明形成该场的电荷分布必然是特殊的

马后炮的说，这个拉氏量是普遍适用的，所以我们的结论暗示了在这一变化下 A_μ 的形式必须变化. $\Rightarrow A_\mu$ 是洛伦兹协变的.

1. 能量：$E=\frac{1}{2}m_1\dot{\vec{r}}_1^2+\frac{1}{2}m_2\dot{\vec{r}}_2^2+V(|\vec{r}_2-\vec{r}_1|)$

动量：空间整体平移对应的守恒量 $\vec{r}_1'=\vec{r}_1+\epsilon\vec{y} \qquad \vec{r}_2'=\vec{r}_2+\epsilon\vec{y}$

$\mathcal{L}'=\frac{1}{2}m_1\dot{\vec{r}}_1^2+\frac{1}{2}m_2\dot{\vec{r}}_2^2+V(|\vec{r}_1+\epsilon\vec{y}-\vec{r}_2-\epsilon\vec{y}|)=\mathcal{L}$ 故而该对称性存在.

$Q=\frac{\partial\mathcal{L}}{\partial\dot{q}^\mu}\Delta q^\mu - f = \vec{y}\cdot(m_1\dot{\vec{r}}_1+m_2\dot{\vec{r}}_2)$ 由 $\vec{y}$ 的任意性，真正的守恒量为：

$\vec{p}=m_1\dot{\vec{r}}_1+m_2\dot{\vec{r}}_2$

角动量：无需计算，旋转变换保持长度，而 $\dot{\vec{r}}_1^2=(\frac{d}{dt}|\vec{r}_1|)^2 \quad \dot{\vec{r}}_2^2=(\frac{d}{dt}|\vec{r}_2|)^2$

故拉氏量显然不变！

$\vec{r}_1'=\vec{r}_1+\epsilon R\vec{r}_1 \qquad \vec{r}_2'=\vec{r}_2+\epsilon R\vec{r}_2 \qquad R_{ij}=-R_{ji}$ 可以把 R_{ij} 写作 $-\epsilon_{ijk}\omega_k$.

$Q=\frac{\partial\mathcal{L}}{\partial\dot{q}^\mu}\Delta q^\mu - f = m_1\dot{x}^i_{(1)}R_{ij}x^j_{(1)}+m_2\dot{x}^i_{(2)}R_{ij}x^j_{(2)} = R_{ij}P^i_{(1)}x^j_{(1)}+R_{ij}P^i_{(2)}x^j_{(2)}$

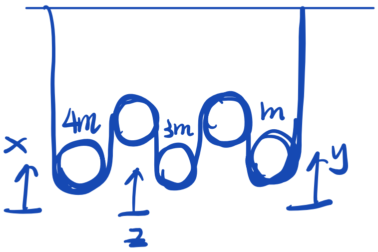
$=\omega_k[\epsilon_{ijk}x^i_{(1)}P^j_{(1)}+\epsilon_{ijk}x^i_{(2)}P^j_{(2)}]$

ω_k 有任意性，因此最终的守恒量为 $L_i=\epsilon_{ijk}x^j_{(1)}P^k_{(1)}+\epsilon_{ijk}x^j_{(2)}P^k_{(2)}$

即 $\vec{L}=\vec{r}_1\times\vec{p}_1+\vec{r}_2\times\vec{p}_2$.

2. 绳长约束：$dz=-dx-dy$.

$$\mathcal{L}=\frac{1}{2}\cdot 4m\dot{x}^2+\frac{1}{2}\cdot m\dot{y}^2+\frac{1}{2}\cdot 3m\dot{z}^2-mg(4x+y+3z)$$



$= \frac{1}{2}m[4\dot{x}^2+\dot{y}^2+3\dot{x}^2+3\dot{y}^2+6\dot{x}\dot{y}] - mg[4x+y-3x-3y]$

$= \frac{1}{2}m(7\dot{x}^2+4\dot{y}^2+6\dot{x}\dot{y}) + mg(x-2y)$

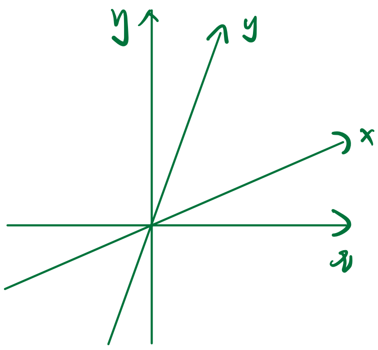
能量守恒：$E=\frac{\partial L}{\partial \dot{x}}\dot{x}+\frac{\partial L}{\partial \dot{y}}\dot{y}-L$.

"动量"守恒：$x\to x+2a\epsilon$ $y\to y+\epsilon a$ $\epsilon\ll 1$ 时，L 不变.

$Q=(7m\dot{x}+3m\dot{y})2a\epsilon+(4m\dot{y}+3m\dot{x})a\epsilon=\epsilon a m(17\dot{x}+10\dot{y})$

$\tilde{p}=m(17\dot{x}+10\dot{y})$

交叉融合的坐标带来交叉项，正交化后会呈现出动量结构！



$\xi=x-2y$ $\eta=x+hy$

$\alpha\dot{\xi}^2+\beta\dot{\eta}^2=7\dot{x}^2+4\dot{y}^2+6\dot{x}\dot{y}$

$(\alpha+\beta)\dot{x}^2+(4\alpha+h^2\beta)\dot{y}^2+(-4\alpha+2h\beta)\dot{x}\dot{y}=7\dot{x}^2+4\dot{y}^2+6\dot{x}\dot{y}$

$\begin{cases}\alpha+\beta=7\\4\alpha+h^2\beta=4\\-4\alpha+2h\beta=6\end{cases} \Rightarrow h^2\beta+2h\beta=10 \Rightarrow \beta=\frac{10}{h^2+2h}$ 由 $\beta=7 \Rightarrow \alpha=\frac{7h^2+14h-10}{h^2+2h}$

$\frac{28h^2+56h-40}{h^2+2h}+\frac{10h^2}{h^2+2h}=4$

$\Rightarrow 38h^2+56h-40=4h^2+8h \Rightarrow 34h^2+48h-40=0$ $h_1=\frac{10}{17}$ $h_2=-\frac{1}{2}$

剪切，可以.

垂直，或旋转，不可！

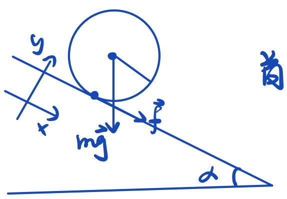
新坐标：$\xi=x-2y$ $\eta=17x+10y$ 可以正交化并得到一个"动量"！

HW4

1. 不懂 D'Alembert 原理
2. 混乱的基
3. 不懂只有独立变分才可以让括号为0.

2. 达朗贝尔原理要求：$\sum_i (\vec{F}_i - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i = 0$

体系只有一个自由度，最后的方程肯定会归为一个。



首先我们考虑刚体的约束，定有：$\begin{cases} |\vec{r}_i - \vec{r}_1| = const \\ |\vec{r}_2 - \vec{r}_3| = const \\ \cdots\cdots \\ |\vec{r}_{n-1} - \vec{r}_n| = const \end{cases}$

那么完全可以选刚体中的一个点，其他点相对于该点只能旋转，且旋转的角度大小不相同。

为方便取质心为这一参考点，质心位置为 (x_0, R)

则 $\delta\vec{r}_i = (\delta x_0 - r_i\sin\theta_i\delta\theta,\ r_i\cos\theta_i\delta\theta)$ $\quad \vec{F}_i = (F_{ix},\ F_{iy})$

再加入纯滚动约束，则有 $\delta x_0 = R\delta\theta$.

则有：

$$\sum_i (F_{ix} - m_i a_{ix})[(R - r_i\sin\theta_i)\delta\theta] + (F_{iy} - m_i a_{iy})(r_i\cos\theta_i\delta\theta) = 0$$

$$\Rightarrow \left[\sum_i (F_{ix} - m_i a_{ix})(R - r_i\sin\theta_i) + (F_{iy} - m_i a_{iy}) r_i\cos\theta_i\right]\delta\theta = 0$$

$$\Rightarrow \sum_i (F_{ix} - m_i a_{ix})(R - r_i\sin\theta_i) + (F_{iy} - m_i a_{iy}) r_i\cos\theta_i = 0$$

代入

$a_{ix} = R\ddot\theta + r_i(\sin\theta_i\ddot\theta + \cos\theta_i\dot\theta^2)$ $\qquad a_{iy} = r_i(-\sin\theta_i\dot\theta^2 + \cos\theta_i\ddot\theta)$

有：

$$\sum_i (F_{ix}R - F_{ix}r_i\sin\theta_i + F_{iy}r_i\cos\theta_i) - \sum_i m_i r_i^2\ddot\theta - \sum_i m_i R^2\ddot\theta + \sum_i m_i R r_i\cos\theta_i\dot\theta^2 = 0$$

先看最后一项：$\sum_i m_i R r_i\cos\theta_i\dot\theta^2 = \int dm\, Rr\cos\theta\,\dot\theta^2 = R\dot\theta^2\int r\cos\theta\, dm = 0$

再看中间项：$\sum_i m_i r_i^2\ddot\theta = I\ddot\theta = \frac{1}{2}MR^2\ddot\theta \Rightarrow \sum_i m_i r_i^2\ddot\theta + \sum_i m_i R^2\ddot\theta = \frac{3}{2}MR^2\ddot\theta$

最后看第一项. 根据约束，摩擦力不应作为主动力（相应的质点不能动），而刚体中内力全部消去，故而只有重力是主动力.

$$\int dm\,(g\sin\alpha R - g\sin\alpha\, r\sin\theta + g\cos\alpha\, r\cos\theta) = \sigma g\int d\theta\, dr\,(rR\sin\alpha - r^2\sin\theta\sin\alpha + r^2\cos\theta\cos\alpha)$$

$= \sigma g\pi R^3\sin\alpha = MgR\sin\alpha$.

则 $\frac{3}{2}MR^2\ddot\theta = MgR\sin\alpha \Rightarrow \ddot\theta = \frac{2}{3}\frac{g}{R}\sin\alpha$.

那么：$\begin{cases} \theta = \frac{1}{3}\frac{gt^2}{R}\sin\alpha + \omega_0 t + \theta_0 \end{cases}$

$x_0 = \frac{1}{2} g t^2 \sin\alpha + v_0 t + x_{0,\ldots}$

这足以描述任意一个质量(元)的运动.

1.

$$E = -mgh + [2\pi h\tan\alpha - l]^2 \cdot \frac{1}{2}k.$$

平衡时有: $\delta E = \frac{\partial E}{\partial h}\delta h$，取 $\delta E = 0 \Leftrightarrow$ 取 $\frac{\partial E}{\partial h} = 0$.

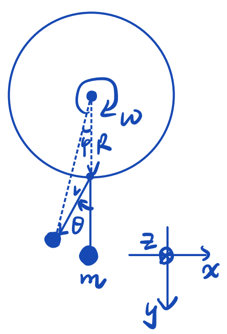
得: $h = \frac{1}{(2\pi\tan\alpha)^2}\frac{mg}{k} + \frac{l}{2\pi\tan\alpha}$

Problems:

1. 个别不用虚功原理，受力分析
2. 忘记原长 l，$V = \frac{1}{2}kh^2\tan^2\alpha$
3. 带动能进去算. $T = \frac{1}{2}m\dot{h}^2$ 肯定不对.

大概一半同学感觉不熟悉这个东西.

3. 取随圆盘共转的非惯性系. 达朗贝尔原理:



$$(F_x - m\ddot{x})\delta x + (F_y - m\ddot{y})\delta y = 0. \quad ①$$

体系受到的主动力为离心惯性力和科氏力.

$$\vec{F}_\omega = -m\omega^2(\vec{R}+\vec{r}) \quad ②$$

$$\vec{F}_c = -2m(\vec{\omega}\times\vec{v}) \quad ③$$

进一步发现 $\vec{F}_c$ 非主动力，因为 $\vec{F}_\omega \perp \vec{v}$，不做虚功. 而离心力中也只有 $\vec{R}$ 一项做虚功，$\vec{r}$ 一项不做功.

$$\begin{cases} x = l\sin\theta \\ y = l(1-\cos\theta) \end{cases} \Rightarrow \begin{cases} \dot{x} = l\cos\theta\,\dot\theta \\ \dot{y} = l\sin\theta\,\dot\theta \end{cases} \Rightarrow \begin{cases} \ddot{x} = l\cos\theta\,\ddot\theta - l\sin\theta\,\dot\theta^2 \\ \ddot{y} = l\sin\theta\,\ddot\theta + l\cos\theta\,\dot\theta^2 \end{cases}$$

代入①式:

$$(-m\ddot{x})\delta x + (m\omega^2 R - m\ddot{y})\delta y = 0 \Rightarrow (-m\ddot{x})\,l\cos\theta\,\delta\theta + (m\omega^2 R - m\ddot{y})\,l\sin\theta\,\delta\theta = 0 \Rightarrow \ddot\theta + \frac{\omega^2 R}{l}\sin\theta = 0$$

恰似一个 $g = \omega^2 R$ 的重力场.

4.

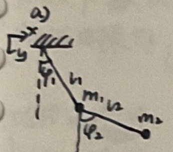
1. 注意坐标，有同学坐标错了.
2. 给定条件问题，只对感兴趣的东西变分.
3. $\hat{e}_\varphi$ 能不能转换成 x, y 坐标未知，要会处理这种问题.
4. 有同学用小角度近似，很棒.

$1dx + 1dy$

$\nabla\times\vec{F} = \partial_1 F_2 - \partial_2 F_1 = \gamma - \beta$

第六次作业

a)



$T_1 = \frac{1}{2} m_1 l_1^2 \dot\varphi_1^2$ ①

$V = -m_1 g l_1 \cos\varphi_1 - m_2 g l_1 \cos\varphi_1 - m_2 g l_2 \cos\varphi_2$ ②

而又有 $\begin{cases} x_2 = l_1\sin\varphi_1 + l_2\sin\varphi_2 \\ y_2 = l_1\cos\varphi_1 + l_2\cos\varphi_2 \end{cases}$ ③

则

$T_2 = \frac{1}{2}m_2(\dot x^2 + \dot y^2) = \frac{1}{2}m_2[l_1^2\dot\varphi_1^2 + l_2^2\dot\varphi_2^2] + 2l_1l_2(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2)\dot\varphi_1\dot\varphi_2]$

$= \frac{1}{2}m_2(l_1^2\dot\varphi_1^2 + l_2^2\dot\varphi_2^2) + m_2 l_1\dot\varphi_1 l_2\dot\varphi_2\cos(\varphi_1-\varphi_2)$ ④

则

$L = \frac{1}{2}m_1 l_1^2\dot\varphi_1^2 + \frac{1}{2}m_2(l_1^2\dot\varphi_1^2 + l_2^2\dot\varphi_2^2) + m_2 l_1\dot\varphi_1 l_2\dot\varphi_2\cos(\varphi_1-\varphi_2) + (m_1+m_2)gl_1\cos\varphi_1 + m_2 g l_2\cos\varphi_2$ ⑤

则：

$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi_1} - \frac{\partial L}{\partial\varphi_1} = 0$ ⑥

得：

$(m_1l_1^2 + m_2l_1^2)\ddot\varphi_1 + m_2l_1l_2[\ddot\varphi_2\cos(\varphi_1-\varphi_2) - \dot\varphi_2\sin(\varphi_1-\varphi_2)\dot\varphi_2(\dot\varphi_1-\dot\varphi_2)] + (m_1+m_2)gl_1\sin\varphi_1 = 0$ ⑦

$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi_2} - \frac{\partial L}{\partial\varphi_2} = 0$ ⑧

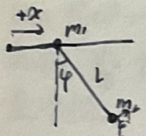
得：

$m_2l_2^2\ddot\varphi_2 + m_2l_1l_2[\ddot\varphi_1\cos(\varphi_1-\varphi_2) - \dot\varphi_1(\dot\varphi_1-\dot\varphi_2)\sin(\varphi_1-\varphi_2)] + m_2gl_2\sin\varphi_2 = 0$ ⑤

其中 ⑦ ⑨ 为运动方程组.

b) 选 m_1 的位置 x 和夹角 φ 为广义坐标，恰体系自由度为 2.

$T = \frac{1}{2}m_1\dot x^2 + \frac{1}{2}m_2[(\dot x + l\dot\varphi\cos\varphi)^2 + (l\dot\varphi\sin\varphi)^2]$ ①

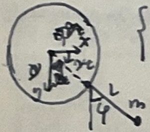


$V = -mgl\cos\varphi$ ②

故 $L = T - V = \frac{1}{2}m_1\dot x^2 + \frac{1}{2}m_2[(\dot x + l\dot\varphi\cos\varphi)^2 + (l\dot\varphi\sin\varphi)^2] + m_2 gl\cos\varphi$. ③

c)

(i) 取 φ 为广义坐标

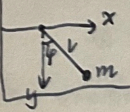


$\begin{cases} x = a\cos\gamma t + l\sin\varphi \\ y = a\sin\gamma t + l\cos\varphi \end{cases}$ ①

则 $T = \frac{1}{2}m(\dot x^2 + \dot y^2) = \frac{1}{2}m[a^2\gamma^2 + l^2\dot\varphi^2 + 2a\gamma l\dot\varphi(-\sin\gamma t\cos\varphi + \cos\gamma t\sin\varphi)]$

$= \frac{1}{2}m[a^2\gamma^2 + l^2\dot\varphi^2 + 2a\gamma\sin(\varphi - \gamma t)]$. ②

~~(ii) 仍取 φ 为广义坐标~~

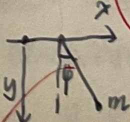


~~$\begin{cases} x = a\cos\gamma t + l\sin\varphi \\ y = l\cos\varphi \end{cases}$ ②~~

而又 $V = -mgy = -mg(a\sin\gamma t + l\cos\varphi)$ ③

$L = \frac{1}{2}m[a^2\gamma^2 + l^2\dot\varphi^2 + 2a\gamma\sin(\varphi - \gamma t)] + mg(a\sin\gamma t + l\cos\varphi)$. ④

(ii) 仍取 φ 为广义坐标.



$\begin{cases} x = a\cos\gamma t + l\sin\varphi \\ y = l\cos\varphi \end{cases}$ ①

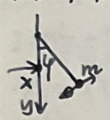
则 $T = \frac{1}{2}m(\dot x^2 + \dot y^2) = \frac{1}{2}m(-a\gamma\sin\gamma t + \dot\varphi l\sin\varphi)^2 + \frac{1}{2}ml^2\sin^2\varphi\,\dot\varphi^2$ ②

而又 $V = -mgl\cos\varphi$ ③

得：

$L = \frac{1}{2}m(-a\gamma\sin\gamma t + \dot\varphi l\sin\varphi)^2 + \frac{1}{2}ml^2\sin^2\varphi\,\dot\varphi^2 + mgl\cos\varphi$ ④

(iii) 仍然取φ为广义坐标.



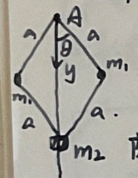
$$\begin{cases} x = l\sin\varphi \\ y = a\cos\gamma t + l\cos\varphi \end{cases} \quad ①$$

$$T = \frac{1}{2}ml^2\dot{\varphi}^2\cos^2\varphi + \frac{1}{2}m(-\gamma a\sin\gamma t + \dot{\varphi}l\sin\varphi)^2 \quad ②$$

$$V = mg(a\cos\gamma t - l\cos\varphi) \quad ③$$

$$L = T - V = \frac{1}{2}ml^2\dot{\varphi}^2\cos^2\varphi + \frac{1}{2}m(-\gamma a\sin\gamma t + \dot{\varphi}l\sin\varphi)^2$$
$$- mg(a\cos\gamma t - l\cos\varphi) \quad ④.$$

d) 由体系对称性，不妨直接取θ为广义坐标



$$\begin{cases} y_2 = 2a\cos\theta \\ y_1 = a\cos\theta \\ r_1 = a\sin\theta \end{cases} \quad ①$$

故而：

$$T_2 = \frac{1}{2}m_2\dot{y}_2^2 = 2m_2a^2\dot{\theta}^2\sin^2\theta \quad ②$$

$$T_1 = m_1(\dot{y}_1^2 + \dot{r}_1^2 + r_1^2\Omega^2) = m_1(a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\theta}^2\cos^2\theta + \Omega^2a^2\sin^2\theta) = m_1(a^2\dot{\theta}^2 + a^2\Omega^2\sin^2\theta)$$
$$= m_1a^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) \quad ③$$

$$V = -2m_1gy_1 - m_2gy_2 = -2m_1ga\cos\theta - 2m_2ag\cos\theta$$
$$= -2ag\cos\theta(m_1 + m_2) \quad ④$$

故

$$L = T - V = 2m_2a^2\dot{\theta}^2\sin^2\theta + m_1a^2(\dot{\theta}^2 + \Omega^2\sin^2\theta)$$
$$+ 2(m_1 + m_2)ag\cos\theta \quad ⑤$$

则有：

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \quad ⑥$$

得：

$$2(2m_2 + m_1)a^2\ddot{\theta} + 4m_2a^2\dot{\theta}^2\sin\theta\cos\theta + 2(m_1 + m_2)$$
$$ag\sin\theta + 2m_1a^2\Omega^2\sin\theta\cos\theta = 0 \quad ⑦$$