另一种等价的诺特定理的证明方式.

$\delta L = \frac{d}{d\epsilon} L(q(\epsilon), \dot{q}(\epsilon), t)$

不如取最简单的变换: $q^\mu \to q'^\mu = q^\mu + \epsilon \Delta q^\mu$ $\epsilon \ll 1$, 为常数.

下证: 若 $\delta L = \epsilon \frac{df}{dt}$, 则存在诺特荷.

略去 ϵ^2 项以上的高阶量, 有:

$$\delta L = L(q') - L(q) = \frac{\partial L}{\partial q^\mu} \epsilon \Delta q^\mu + \frac{\partial L}{\partial \dot{q}^\mu} \epsilon \frac{d}{dt}(\Delta q^\mu)$$

$$= \epsilon \frac{\partial L}{\partial q^\mu} \Delta q^\mu + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu} \epsilon \Delta q^\mu\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu}\right) \epsilon \Delta q^\mu$$

$$= \epsilon \left[\frac{\partial L}{\partial q^\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu}\right)\right] \Delta q^\mu + \epsilon \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu} \Delta q^\mu\right)$$

根据 E-L 方程: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu}\right) - \frac{\partial L}{\partial q^\mu} = 0$

故: $\delta L = \epsilon \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu} \Delta q^\mu\right)$

根据我们的假设: $\delta L = \epsilon \frac{df}{dt}$, 故:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu} \Delta q^\mu - f\right) = 0.$

此即在真实的运动中:

$Q = \frac{\partial L}{\partial \dot{q}^\mu} \Delta q^\mu - f$

是一个常数, 即守恒. Q.E.D.

下面给出几个常见的 Δq.

(1) 空间平移: $\Delta q^\mu = \alpha^\mu$, α^μ 为常数.

(2) 时间平移: $\Delta q^\mu = \dot{q}^\mu$

(3) 空间转动：$\Delta q^\mu = X^\mu{}_\nu q^\nu \quad X^\mu{}_\nu = -X^\nu{}_\mu$.

证明：$R^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon X^\mu{}_\nu \quad R^\mu{}_\nu (R^T)^\nu{}_\rho = \delta^\mu{}_\rho$.

$R^\mu{}_\nu (R^T)^\nu{}_\rho = (\delta^\mu{}_\nu + \epsilon X^\mu{}_\nu)(\delta^{T\nu}{}_\rho + \epsilon X^{T\nu}{}_\rho)$，$\delta^{T\nu}{}_\rho = \delta^\nu{}_\rho$.

$\Rightarrow R^\mu{}_\nu (R^T)^\nu{}_\rho = (\delta^\mu{}_\nu + \epsilon X^\mu{}_\nu)(\delta^\nu{}_\rho + \epsilon X^{T\nu}{}_\rho) = \delta^\mu{}_\rho + \epsilon (X^\mu{}_\rho + X^{T\mu}{}_\rho) + O(\epsilon^2)$

那么在一阶近似上，必有应 $X^\mu{}_\rho = -X^{T\mu}{}_\rho$ 即 $X^\mu{}_\rho = -X^\rho{}_\mu$. 证毕.

其实三维空间旋转，洛伦兹boost都是这样的变换.

计算举例：若 $\frac{\partial L}{\partial t} = 0$，则：

$$\delta L = \frac{\partial L}{\partial q^\mu} \epsilon \dot{q}^\mu + \frac{\partial L}{\partial \dot{q}^\mu} \epsilon \ddot{q}^\mu = \epsilon \left(\frac{\partial L}{\partial q^\mu} \dot{q}^\mu + \frac{\partial L}{\partial \dot{q}^\mu} \ddot{q}^\mu + \frac{\partial L}{\partial t} \right) = \frac{dL}{dt}.$$

则：$Q = \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu - L$ 守恒，此即能量守恒.

感做：请如法炮制给出 (1) (3) 对应的守恒量.