目录

第一章　微分方程举例

本章先给出3个微分方程的例子

例1：$\Delta u = 0$ in $\mathbb{R}^n$ 的径向解

解：由 $\dfrac{\partial u}{\partial x_i} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x_i}$

且 $$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x_i \partial x_j}$$

Step 1： $r^2 = \sum_{i=1}^n x_i^2 \Rightarrow 2r\dfrac{\partial r}{\partial x_i} = 2x_i$

$$\Rightarrow 2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} + 2r\frac{\partial^2 r}{\partial x_i \partial x_j} = 2\delta_{ij}$$

由上面两式知 $\dfrac{\partial r}{\partial x_i} = \dfrac{x_i}{r}$

且有 $$\frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}$$

故有 $$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial r^2}\frac{x_i^2}{r^2} + \frac{\partial u}{\partial r}\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

$$\Rightarrow \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} = 0$$

Step 2 ： 当 $n=2$ 时　$u'' + \dfrac{1}{r}u' = 0$

即有 $(ru')' = 0 \Rightarrow ru' = C_0 \Rightarrow u' = \dfrac{C_0}{r}$

$$\Rightarrow u = c_0 \ln r + c_1$$

当 $n \geq 3$ 时　$ru'' + (n-1)u' = 0$

令 $v = ru'$　则 $v' = u' + ru''$.

从而 $v' + \dfrac{n-2}{r}v = 0 \Rightarrow (\ln v)' = \dfrac{2-n}{r}$

$$\Rightarrow \ln v = (2-n)\ln r + C_0 \Rightarrow v = c\,r^{2-n}$$

$$\Rightarrow u' = c\,r^{1-n} \Rightarrow u = c\,r^{2-n}.$$

例2：$y''(t)+\lambda y(t)=0$ in S^1

其中 S^1 为 $\{(x,y)\mid x^2+y^2=1\}$.

解：同乘 y 得 $yy''+\lambda y^2=0$

$$\Rightarrow \int_{S^1} yy''+\lambda\int_{S^1} y^2=0$$

$$\Rightarrow yy'\Big|_{\partial S^1}-\int_{S^1}(y')^2+\lambda\int_{S^1}y^2=0.$$

由 $\partial S^1=\phi$. 故 $\lambda=\dfrac{\int_{S^1}(y')^2}{\int_S y^2}>0$.

特别的，若 $\lambda=0$，则 $y''\equiv 0$.

$y(t)=at+b$. 结合 y 的定义知 $y\equiv b$.

例3：$\begin{cases} y''+\lambda y=0 & \text{in } [0,1] \\ y(0)=y(1)=0 \end{cases}$

解：同上有 $\lambda\geq 0$. 考虑 $y(t)=\sin(\sqrt{\lambda}t)$

易见 $y''+\lambda y=-\lambda\sin\sqrt{\lambda}t+\lambda\sin\sqrt{\lambda}t=0$

由 $y(0)=y(1)=0 \Rightarrow \sin\sqrt{\lambda}=0$

$\Rightarrow \sqrt{\lambda}=k\pi \Rightarrow \lambda=k^2\pi^2 \quad (k\in N)$

称 λ 为方程在区域 $[0,1]$ 上的特征值.

之后的课程将详细讨论各类微分方程.

第二章 初等积分法

§2.1 恰当方程（曲线积分）

考虑方程 $P(x,y)\,dx + Q(x,y)\,dy = 0 \quad \cdots (*)$

若存在函数 Φ s.t. $\dfrac{\partial \Phi}{\partial x} = P$，$\dfrac{\partial \Phi}{\partial y} = Q$

则称其为恰当方程

例1：$(2x\sin y + 3x^2 y)\,dx + (x^3 + x^2\cos y + y^2)\,dy = 0$

易见 $\dfrac{\partial P}{\partial y} = 2x\cos y + 3x^2 = \dfrac{\partial Q}{\partial x}$

由下述定理知其为恰当方程

定理1：设 $P(x,y)$，$Q(x,y)$ 在 $R: \alpha < x < \beta$，$\gamma < y < \delta$

上连续，且有连续一阶偏导数 $\dfrac{\partial P}{\partial y}$ 与 $\dfrac{\partial Q}{\partial x}$

则：$(*)$ 为恰当方程 $\Longleftrightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

证明：由 $\dfrac{\partial P}{\partial y} = \dfrac{\partial^2 \Phi}{\partial y \partial x} = \dfrac{\partial Q}{\partial x}$ 知必要性显然

另一方面，取 $\Phi = \int_{x_0}^{x} P\,dx + \psi(y)$ ψ 待定

则 $\dfrac{\partial \Phi}{\partial y} = \int_{x_0}^{x} \dfrac{\partial P}{\partial y}(x,y)\,dx + \psi'(y)$

$= \int_{x_0}^{x} \dfrac{\partial Q}{\partial x}\,dx + \psi'(y) = Q(x,y) - Q(x_0,y) + \psi'(y)$

取 $\psi(y) = \int_{y_0}^{y} Q(x_0,y)\,dy$ 即可

这样构造了 $\Phi = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy$

满足题意，这就证明了充分性

回到例1，$\Phi = \int_{x_0}^{x} (2t\sin y + 3t^2 y)\,dt + \psi(y)$

$= x^2\sin y + x^3 y + \psi(y)$

$$\frac{\partial \Phi}{\partial y} = Q \Rightarrow x^2\cos y + x^3 + \varphi'(y) = x^3 + x^2\cos y + y^2$$

$$\Rightarrow \varphi(y) = \frac{y^3}{3} + C \text{，故 } \Phi = x^2\sin y + x^3 y + \frac{y^3}{3} + C$$

§2.2 分离变量法

考虑方程 $\frac{dy}{dx} = -p(x)y$.

$$\Rightarrow \frac{dy}{y} = -p(x)\,dx \Rightarrow \ln y = -\int_{x_0}^{x} p(t)\,dt + C$$

$$\Rightarrow y = Ce^{-\int p(x)dx}$$

§2.3 常数变易法

考虑 $\frac{dy}{dx} + p(x)y = q(x)$.

令 $y(x) = c(x)e^{-\int p(x)dx}$

则有 $y'(x) = c'(x)e^{-\int p(x)dx} - c(x)p(x)e^{-\int p(x)dx}$

代入原方程有 $c'(x)e^{-\int p(x)dx} = q(x)$

故有 $c'(x) = q(x)e^{\int p(x)dx}$

$$c(x) = \int q(x)e^{\int p(x)dx}dx + C$$

$$y = e^{-\int p(x)dx}\left(C + \int q(x)e^{\int p(x)dx}\right)$$

为解的表达式

例1： $y' + \frac{y}{x} = x^3 \quad (x>0)$

解： step 1 ： 考虑 $y' + \frac{y}{x} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$

$\Rightarrow \ln y = -\ln x + C \Rightarrow y = \frac{c}{x}$

Step 2. 令 $y = \frac{c(x)}{x}$ 代入原方程

则有 $y'(x) = \frac{c'(x)}{x} - \frac{c(x)}{x^2}$

$y' + \frac{y}{x} = \frac{c'(x)}{x} = x^3 \Rightarrow c'(x) = x^4$.

$\Rightarrow c(x) = \frac{x^5}{5} + C_0$. 代入有

$y = \frac{c(x)}{x} = \frac{x^4}{5} + \frac{C_0}{x}$ 代入验证是解.

§2.4 初等变换法

例1: $\frac{dy}{dx} = f(x+y)$ 令 $u = x+y$.

得 $\frac{du}{dx} = 1 + \frac{dy}{dx} = 1 + f(u)$

$\Rightarrow \frac{du}{1+f(u)} = dx \Rightarrow \int \frac{du}{1+f(u)} = x + C$

积分可求得原方程的解.

齐次方程: 若 $P(x,y)dx + Q(x,y)dy = 0 \cdots (*)$

其中 P. Q 满足 $P(tx,ty) = t^m P(x,y)$

$Q(tx,ty) = t^m Q(x,y)$. $(m \in N)$

则称方程 (*) 为齐次方程.

例2: $\frac{dy}{dx} = \frac{x+y}{x-y}$.

解: 令 $u = \frac{y}{x}$. 则 $\frac{dy}{dx} = \frac{d(ux)}{dx} = \frac{du}{dx} \cdot x + u$

又有 $\frac{x+y}{x-y} = \frac{1+u}{1-u}$

从而有 $x\frac{du}{dx}+u=\frac{1+u}{1-u}$.

$$\Rightarrow x\frac{du}{dx}=\frac{1+u^2}{1-u^2} \Rightarrow \frac{1-u}{1+u^2}du=\frac{dx}{x}$$

$$\Rightarrow \int \frac{1-u}{1+u^2}du=\int\frac{dx}{x}=\ln|x|+c.$$

$$\Rightarrow \arctan u-\frac{1}{2}\ln(1+u^2)=\ln|x|+c$$

$$\Rightarrow |x|\sqrt{1+u^2}=ce^{\arctan u}$$

$\Rightarrow \sqrt{x^2+y^2}=ce^{\arctan\frac{y}{x}}$ 为通积分.

§2.5 积分因子法

若 $P(x,y)dx+Q(x,y)dy=0$ 为恰当方程.

则通积分为 $\int_{x_0}^{x}P(x,y)dx+\int_{y_0}^{y}Q(x_0,y)dy=C$

第§2.2, §2.3, §2.4 节说明了如何化为恰当方程.

变量分离 $X(x)Y_1(y)dx+X_1(x)Y(y)dy=0$.

定义 $\mu(x,y)=\frac{1}{X_1(x)Y_1(y)}$ 同乘两边.

则有 $\frac{X(x)}{X_1(x)}dx+\frac{Y(y)}{Y_1(y)}dy=0$.

例：一阶线性方程 $\frac{dy}{dx}+p(x)y=q(x)$.

取 $\mu(x,y)=e^{\int p(x)dx}$ 乘两边. 则有

$$y'e^{\int p(x)dx}+p(x)ye^{\int p(x)dx}=q(x)e^{\int p(x)dx}$$

$$\Rightarrow d\left(ye^{\int p(x)dx}\right)=q(x)e^{\int p(x)dx}dx$$

从而 $y=e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx\right)$

目标：找一个 $\mu(x,y)$ 称为积分因子

s.t. $\mu(x,y)P(x,y)dx+\mu(x,y)Q(x,y)dy=0$ 为恰当方程

即 $\dfrac{\partial(\mu P)}{\partial y}=\dfrac{\partial(\mu Q)}{\partial x}$

$\Leftrightarrow \dfrac{\partial\mu}{\partial y}P+\mu\dfrac{\partial P}{\partial y}=\dfrac{\partial\mu}{\partial x}Q+\mu\dfrac{\partial Q}{\partial x}$

$\Leftrightarrow P\dfrac{\partial\mu}{\partial y}-Q\dfrac{\partial\mu}{\partial x}=\mu\left(\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)$

某些特殊情况可找到 μ. 例如 $\mu=\mu(x)$ 时

$\dfrac{\partial\mu}{\partial y}=0$ 有 $\dfrac{d\mu}{dx}=-\dfrac{1}{Q}\cdot\mu\left(\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)$

即 $\dfrac{1}{\mu}\dfrac{d\mu}{dx}=\dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)$

故 $Pdx+Qdy=0$ 有一个只依赖于 x 的 μ

的必要条件为 $\dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)$ 只依赖于 x

反之. 若 $G(x)\triangleq\dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)$

由 $\dfrac{1}{\mu}\dfrac{d\mu}{dx}=G \Rightarrow \mu(x)=e^{\int G(x)dx}$ 为积分因子.

综上得到：方程 $Pdx+Qdy=0$ 有一个只依赖于 x

的积分因子的充要条件为

$\dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)$ 只依赖于 x. 若记之为 $G(x)$

则 $\mu(x)=e^{\int G(x)dx}$ 为原方程积分因子.

例1：$(3x^3+y)dx+(2x^2y-x)dy=0$

解法一：$\dfrac{\partial P}{\partial y}=1 \quad \dfrac{\partial Q}{\partial x}=4xy-1$

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{1}{2x^2y-x}\left(2-4xy\right)=-\frac{2}{x}$$

$$\mu(x,y)=e^{-\int\frac{2}{x}dx}=\frac{1}{x^2}$$ 同乘两边

$$3x\,dx+2y\,dy+\frac{y\,dx-x\,dy}{x^2}=0$$

$$\frac{3}{2}x^2+y^2-\frac{y}{x}=C$$

解法二：$(3x^3+y)dx+(2x^2y-x)dy=0$

$$(3x^3dx+2x^2y\,dy)+(y\,dx-x\,dy)=0$$

积分因子 $\frac{1}{x^2}$ $\qquad \frac{1}{x^2}.\ \frac{1}{y^2}.\ \frac{1}{x^2+y^2}$

公共积分因子为 $\frac{1}{x^2}$. 以下同解法一.

定理 2-6. 若 $\mu=\mu(x,y)$ 为 $P\,dx+Q\,dy=0$ 的积分因子

s.t. $\mu P\,dx+\mu Q\,dy=d\Phi$.

则 $\mu g(\Phi)$ 也为积分因子.

证：$$\frac{\partial(\mu g(\Phi)P)}{\partial y}=\frac{\partial(\mu P)}{\partial y}g(\Phi)+\mu P g'(\Phi)\frac{\partial\Phi}{\partial y}$$

$$=\frac{\partial(\mu Q)}{\partial x}g(\Phi)+\mu Q g'(\Phi)\frac{\partial\Phi}{\partial x}=\frac{\partial(\mu g(\Phi)Q)}{\partial x}$$

分组积分因子：

$$(P_1dx+Q_1dy)+(P_2dx+Q_2dy)=0.$$

则有 若 μ_i 为 $P_i\,dx+Q_i\,dy=0$ 的积分因子.

$d\Phi_i=\mu_i(P_i\,dx+Q_i\,dy)\quad i=1.2$

定理 2-6 $\Longrightarrow$ $\mu_1g_1(\Phi_1)$. $\mu_2g_2(\Phi_2)$ 为积分因子

取适当 $g_1.\ g_2$ s.t. $\mu_1g_1(\Phi_1)=\mu_2g_2(\Phi_2)$

例2 $(x^3y-2y^2)dx+x^4dy=0$.

解：$x^3y\,dx+x^4dy=0$ 有积分因子 x^{-3}.

$y\,dx+x\,dy=d(xy)$.

而 $-2y^2dx$ 有积分因子 y^{-2}.

取 g_1, g_2 s.t. $x^{-3}g_1(xy)=y^{-2}g_2(x)$

取 $g_1(xy)=\dfrac{1}{(xy)^2}$ $\quad g_2(x)=\dfrac{1}{x^5}$.

$$\frac{d(xy)}{(xy)^2}-\frac{2}{x^5}dx=0 \Rightarrow -\frac{1}{xy}+\frac{1}{2x^4}=C$$

$$y=\frac{2x^3}{2Cx^4+1}$$ 易见特解为 $x\equiv0$ 与 $y\equiv0$

最后指出若 $Pdx+Qdy=0$ 为齐次方程.

则 $\mu=\dfrac{1}{xP+yQ}$ 为一个积分因子.

证：$y=ux$. 则 $\mu=\dfrac{1}{x^{m+1}(P+uQ)}$

$$Pdx+Qdy=x^mP(1,u)dx+x^mQ(1,u)d(ux)$$

$$\mu Pdx+\mu Qdy=\frac{P}{x(P+uQ)}dx+\frac{Q}{x(P+uQ)}d(ux)=0$$

$$\Leftrightarrow \frac{dx}{x}+\frac{Q(1,u)\,du}{P(1,u)+uQ(1,u)}=0$$ 为恰当方程.

例3 $(x+y)dx-(x-y)dy=0$.

解：$$\mu=\frac{1}{x(x+y)+y(y-x)}=\frac{1}{x^2+y^2}$$

故化为 $\dfrac{x+y}{x^2+y^2}dx-\dfrac{x-y}{x^2+y^2}dy=0$ 为恰当方程

即 $\frac{1}{2}\ln(x^2+y^2)-\arctan\frac{y}{x}=C$

例4：求出 $u_t = \Delta u$ 的一个解 $(n=1)$

解：设 $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$ 且 $u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$

$\alpha.\beta$ 待定 $(\alpha.\beta > 0)$ v 为待定函数.

设 $\lambda = t^{-1}$，则有 $u(x,t) = \frac{1}{t^\alpha} u\left(\frac{x}{t^\beta}, 1\right)$

故有 $v(y) = u(y,1)$ 定义 $y = t^{-\beta} x$.

$$u_t = -\alpha t^{-\alpha-1} v(y) + t^{-\alpha} v'(y)(-\beta) t^{-\beta-1} x$$

$$u_x = t^{-(\alpha+\beta)} v'(y) \quad u_{xx} = t^{-(\alpha+2\beta)} v''(y)$$

$$u_t = u_{xx} \Rightarrow -\alpha t^{-\alpha-1} v + t^{-\alpha-1} v'(-\beta) y = t^{-(\alpha+2\beta)} v''(y)$$

$$\xRightarrow{令\beta=\frac{1}{2}} \alpha v + \frac{1}{2} v' y = -v'' \xRightarrow{令\alpha=\frac{1}{2}} \frac{v}{2} + \frac{v'}{2} y = -v''$$

$$\Rightarrow -\frac{1}{2}(yv)' = v'' \Rightarrow -\frac{1}{2} yv = v' + c$$

由物理背景知 $c = 0$ 故 $\frac{dv}{dy} = -\frac{1}{2} vy$

$$\Rightarrow \frac{dv}{v} = -\frac{y}{2} dy \Rightarrow v = b e^{-\frac{|y|^2}{4}} = b e^{-\frac{|x|^2}{4t}}$$

一般定义 $H(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$ 为热核.

注：对一般情形 n.

$$u_t = -\alpha t^{-\alpha-1} v(y) + t^{-\alpha} Dv(y)(-\beta) t^{-\beta-1} \cdot x$$

$$Du = t^{-(\alpha+\beta)} Dv\left(\frac{x}{t^\beta}\right) = t^{-(\alpha+\beta)} Dv(y)$$

$$\Delta u = t^{-(\alpha+2\beta)} \Delta v(y)$$

由 $u_t = \Delta u \Rightarrow \alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y Dv(y)$

$$+ t^{-(\alpha+2\beta)} \Delta v(y) = 0$$

取 $\beta=\frac{1}{2}$ 有 $\alpha v+\frac{1}{2}y\cdot Dv+\Delta v=0$ ……(*)

考虑径向解 $v(y)=w(|y|)\triangleq w(r)$.

有 $v_i(y)=w'(|y|)\cdot\frac{y_i}{r}=w'(r)\frac{y_i}{r}$

$v_{ii}(y)=w''(r)\frac{y_i^2}{r^2}+w'(r)\frac{r^2-y_i^2}{r^3}$

$\Rightarrow \Delta v=w''+\frac{n-1}{r}w'$

$(*)\Longleftrightarrow \alpha w+\frac{1}{2}r\cdot w'+w''+\frac{n-1}{r}w'=0$

令 $\alpha=\frac{n}{2}$. 则 $(*)\Longleftrightarrow (r^{n-1}w')'+\frac{1}{2}(r^n w)'=0$

$\Rightarrow r^{n-1}w'+\frac{1}{2}r^n w=C$ 由物理背景知 $C=0$

即 $w'=-\frac{1}{2}rw \Rightarrow \frac{dw}{w}=-\frac{1}{2}r\,dr$

$\Rightarrow w=ce^{-\frac{r^2}{4}} \Rightarrow v(y)=ce^{-\frac{|y|^2}{4}}$

$\Rightarrow u(x)=ce^{-\frac{|x|^2}{4t}}$ 一般取 $c=\frac{1}{(4\pi t)^{\frac{n}{2}}}$

定义 $\Phi(x,t)=\begin{cases}0 & (x\in R^n,\ t\leq 0)\\ \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}} & (x\in R^n,\ t>0)\end{cases}$.

为热方程的基本解.

可验证 $\int_{R^n}\Phi(x,t)\,dx=1$.

第三章 存在唯一性定理

§3.1 Picard存在唯一性定理

设函数 $f(x,y)$ 在D内满足不等式

$$|f(x,y_1)-f(x,y_2)| \le L\,|y_1-y_2| \quad (L>0)$$

则称 f 在D内对 y 满足 Lipschitz 条件.

易见若 f 在凸区域D内对 y 有连续偏导数

且D有界闭，则 f 满足 Lipschitz 条件

定理3.1 设初值问题

$$(E):\ \frac{dy}{dx}=f(x,y) \qquad y(x_0)=y_0$$

$f\in C(R)$ $\quad R: |x-x_0|\le a \quad |y-y_0|\le b$.

f 对 y 满足 Lipschitz 条件，则 (E) 在 $I=[x_0-h,\ x_0+h]$ 上

存在唯一解. 其中 $h=\min\{a,\frac{b}{M}\}$. $M=\max\limits_{(x,y)\in R}|f(x,y)|$

证明：分四步完成.

step 1：$\quad \frac{dy}{dx}=f(x,y) \Longleftrightarrow y=y_0+\int_{x_0}^{x} f(x,y)\,dx$

Step 2：定义 $y_{n+1}(x)\triangleq y_0+\int_{x_0}^{x} f(x,y_n(x))\,dx$

其中 $y_0(x)=y_0$.

当 $n=0$ 时 $\because f(x,y_0(x))\in C(I)$.

$\therefore y_1(x)=y_0+\int_{x_0}^{x} f(x,y_0(x))\,dx \in C^1(I)$

且 $|y_1(x)-y_0|\le\left|\int_{x_0}^{x}|f(x,y_0(x))|\,dx\right|\le M|x-x_0|$

$\le Mh\le b$

$\therefore f(x,y_1(x))\in C(I)$ 从而

$y_2(x)=y_0+\int_{x_0}^{x} f(x,y_1(x))\,dx \quad (x\in I)$

也是连续可微的

且满足 $|y_2(x) - y_0| \le \left|\int_{x_0}^{x} f(x, y_1(x))\,dx\right|$

$\le M|x-x_0| \le Mh \le b \quad (x\in I)$

以此类推 $y_n \in C(I)$ 且 $|y_n - y_0| \le M|x-x_0|$

Step 3：$y_n(x)$ 收敛性等价于 $\sum_{n=1}^{\infty}(y_{n+1}(x) - y_n(x))$

下证其一致收敛。我们归纳证明下式

$$|y_{n+1}(x) - y_n(x)| \le \frac{M}{L}\frac{(L|x-x_0|)^{n+1}}{(n+1)!} \quad \cdots (*)$$

当 $n=0$ 时 即 $|y_1(x) - y_0(x)| \le M|x-x_0|$

设 $n=k$ 时 $(*)$ 式成立，则由

$$|y_{k+2}(x) - y_{k+1}(x)| = \left|\int_{x_0}^{x}[f(x, y_{k+1}(x)) - f(x, y_k(x))]\,dx\right|$$

$$\le \left|\int_{x_0}^{x} L|y_{k+1}(x) - y_k(x)|\,dx\right|$$

$$\le M\left|\int_{x_0}^{x}\frac{(L|x-x_0|)^{k+1}}{(k+1)!}\,dx\right|$$

$$= \frac{M}{L}\frac{(L|x-x_0|)^{k+2}}{(k+2)!}$$ 即 $n=k+1$ 时 $(*)$ 也成立

由 $(*)$ 式 $\Rightarrow |y_{n+p}(x) - y_n(x)| \le \frac{M}{L}\sum_{k=1}^{p-1}\frac{(L|x-x_0|)^{n+k}}{(k+n)!}$

$\le \frac{M}{L}\left(e^{Lh} - \sum_{k=1}^{n}\frac{(Lh)^k}{k!}\right)$ 由柯西收敛准则知

$y_n(x)$ 在 I 上一致收敛，记 $\varphi(x) = \lim_{n\to\infty} y_n(x) \quad (x\in I)$

由一致收敛性知 $\varphi(x) = y_0 + \int_{x_0}^{x} f(x, \varphi(x))\,dx \quad (x\in I)$

Step 4：最后证明唯一性。

设方程有两个解 $y=u(x)$ 和 $y=v(x)$

令 $J = [x_0 - d, x_0 + d]$ 为共同存在区间。

从而 $u(x)-v(x)=\int_{x_0}^{x}[f(x,u(x))-f(x,v(x))]dx$

$\Rightarrow |u-v| \le L\left|\int_{x_0}^{x}|u(x)-v(x)|dx\right| \quad \cdots (**)$

定义 $K=\max_{x\in J}|u(x)-v(x)|$ 则上式 $\le LK|x-x_0|$ 代入(**)

$|u(x)-v(x)| \le L\left|\int_{x_0}^{x}LK|x-x_0|dx\right| = K\frac{L^2|x-x_0|^2}{2}$

再代入(**)，以此类推有 $|u-v| \le K\frac{(L|x-x_0|)^n}{n!}$ $(x\in J)$

令 $n\to\infty$ 得 $u(x)\equiv v(x)$，这就证明了唯一性。

设函数 $f(x,y)$ 在 G 内连续，且满足

$|f(x,y_1)-f(x,y_2)| \le F(|y_1-y_2|)$

其中 $F(r)>0$ 为 r 的连续函数，且

$\int_0^{r_1}\frac{dr}{F(r)}=\infty$，则称 f 在 G 内对 y 满足 Osgood 条件。

易见 $F(r)=Lr$ 时即为 Lipschitz 条件。

定理 3.2. 设 $f(x,y)$ 在 G 内对 y 满足 Osgood 条件，则 $\frac{dy}{dx}=f(x,y)$ 在 G 内过每一点的解唯一。

证明：设 $\exists (x_0,y_0)\in G$ s.t. 原方程有 y_1, y_2 两个解，

且 $\exists x_1>x_0$. s.t. $y_1(x_1)\ne y_2(x_1)$

不妨设 $y_1(x_1)>y_2(x_1)$ 且令 $\bar{x}=\sup_{x_0\le x\le x_1}\{x \mid y_1(x)=y_2(x)\}$

则显然有 $x_0\le \bar{x}<x_1$，且 $r\triangleq y_1-y_2>0$ $(\bar{x}<x\le x_1)$

且有 $r(\bar{x})=0$，故 $r'(x)=y_1'(x)-y_2'(x)$

$=f(x,y_1(x))-f(x,y_2(x)) \le F(|y_1(x)-y_2(x)|)=F(r(x))$

$\Rightarrow \frac{dr}{F(r)} \le dx \quad (\bar{x}<x\le x_1)$

$\Rightarrow \int_0^{r_1}\frac{dr}{F(r)} \le x_1-\bar{x} <+\infty$，与条件矛盾（$\because$ 左边 $=+\infty$）

Laplace 方程解的唯一性.

引理 $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ 则 $u \equiv 0$ in Ω

证明 $\Delta u = 0 \Rightarrow u\Delta u = 0 \Rightarrow \int_\Omega u\Delta u\, dx = 0$

$\Rightarrow \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds - \int_\Omega |\nabla u|^2 dx = 0$

$\Rightarrow \int_\Omega |\nabla u|^2 dx = 0 \Rightarrow \nabla u \equiv 0 \Rightarrow u \equiv C \Rightarrow u \equiv 0$.

定理 $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$ 的解若存在则唯一.

证明: 设 u_1. u_2 均为解. 则有

$\begin{cases} \Delta(u_1 - u_2) = f - f = 0 & \text{in } \Omega \\ u_1 - u_2 = \varphi - \varphi = 0 & \text{on } \partial\Omega \end{cases}$

由引理知 $u_1 \equiv u_2$. 即解唯一.

补充: 若 u 满足 $\begin{cases} \Delta u + c(x) u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

其中 $c(x) \leq 0$. 则 $u \equiv 0$ in Ω

证明: $u\Delta u + c(x) u^2 = 0 \Rightarrow \int_\Omega u\Delta u + \int_\Omega c(x) u^2 = 0$

$\Rightarrow \int_\Omega |\nabla u|^2 dx = \int_\Omega c(x) u^2 dx$.

$\because c(x) \leq 0$. 故有 $\int_\Omega |\nabla u|^2 dx \leq 0$.

从而 $|\nabla u| \equiv 0$ 即 $u \equiv C$ 结合 $u = 0$ on $\partial\Omega$

得 $u \equiv 0$ in Ω

§3.2 Peano 存在性定理

本节推广 Picard 存在唯一性. 将 Lipschitz 条件换成 $f(x,y)$ 连续. 则解存在

欧拉折线法

设 $\frac{dy}{dx}=f(x,y)$，$y(x_0)=y_0$

R: $|x-x_0|\le a$，$|y-y_0|\le b$ $f\in C(R)$

$M\triangleq\sup\limits_{(x,y)\in R}|f(x,y)|$ $h\triangleq\min\{a,\frac{b}{M}\}$.

则若 $y(x)$ 为 $|x-x_0|\le h$ 上的解，有 $|y-y_0|\le b$.

将 (x_0-h, x_0+h) $2n$ 等分，每份长度 $h_n=\frac{h}{n}$.

从 $P_0(x_0,y_0)$ 出发向右作折线

$y_1=y_0+f(x_0,y_0)(x_1-x_0)$ $y_2=y_0+f(x_1,y_1)(x_2-x_1)$ y_1

$\cdots$ $y_n=y_{n-1}+f(x_{n-1},y_{n-1})(x_n-x_{n-1})$

记 $\varphi_n(x)$ 为所作的欧拉折线，易知 $x\in(x_s, x_{s+1}]$ 时

$\varphi_n(x)=y_s+f(x_s,y_s)(x-x_s)$

$=y_{s-1}+f(x_{s-1},y_{s-1})(x_s-x_{s-1})+f(x_s,y_s)(x-x_s)$

$=\cdots=y_0+\sum\limits_{k=0}^{s-1}f(x_k,y_k)(x_{k+1}-x_k)+f(x_s,y_s)(x-x_s)$

引理3.1 $\varphi_n(x)$ 在 (x_0-h, x_0+h) 上有一个一致收敛子列

证明: $|\varphi_n(x)-y_0|\le\left|\sum\limits_{k=0}^{s-1}f(x_k,y_k)(x_{k+1}-x_k)\right|$

$+|f(x_s,y_s)(x-x_s)|\le M\left[\sum\limits_{k=0}^{s-1}(x_{k+1}-x_k)+x-x_s\right]$

$=M(x-x_0)\le Mh\le b$，故一致有界.

又由构造知 φ_n 每一段斜率在 $-M, M$ 之间.

从而 $|\varphi_n(s)-\varphi_n(t)|\le M|s-t|$

$\Rightarrow$ φ_n 等度连续.

由 Arzela-Ascoli 引理知结论成立.

引理 3.2　$\varphi_n(x)$ 在 $|x-x_0|\le h$ 上满足

$\varphi_n(x)=y_0+\int_{x_0}^{x} f(x,\varphi_n(x))\,dx+d_n(x)$，其中 $\lim\limits_{n\to\infty} d_n(x)=0$

分析：① 第一段折线

$$\varphi_n(x)=y_0+\int_{x_0}^{x} f(x_0,y_0)\,dx=y_0+\int_{x_0}^{x} f(x,\varphi_n(x))\,dx$$

$$+\int_{x_0}^{x}[f(x_0,y_0)-f(x,\varphi_n(x))]\,dx$$

定义 $d_n(x)=\int_{x_0}^{x}[f(x_0,y_0)-f(x,\varphi_n(x))]\,dx$

则由 $f(x,y)$ 的连续性知 $\forall\varepsilon>0$，$\exists N\in N^*$，$n>N$ 时

$$|f(x_0,y_0)-f(x,\varphi_n(x))|<\frac{\varepsilon}{h}$$

故 $|d_n(x)|\le\int_{x_0}^{x}\frac{\varepsilon}{h}\,dx\le\frac{\varepsilon}{h}(x_1-x_0)=\frac{\varepsilon}{h}\cdot h_n=\frac{\varepsilon}{n}$.

② 第二段折线

$$\varphi_n(x)=y_1+\int_{x_1}^{x} f(x_1,y_1)\,dx=y_0+\int_{x_0}^{x_1} f(x_0,y_0)\,dx$$

$$+\int_{x_1}^{x} f(x_1,y_1)\,dx=y_0+\int_{x_0}^{x_1} f(x,\varphi_n(x))\,dx+\int_{x_1}^{x} f(x,\varphi_n(x))\,dx$$

$$+\int_{x_0}^{x_1}[f(x_0,y_0)-f(x,\varphi_n(x))]\,dx+\int_{x_1}^{x}[f(x_1,y_1)-f(x,\varphi_n(x))]\,dx$$

$$\triangleq y_0+\int_{x_0}^{x} f(x,\varphi_n(x))\,dx+d_n(1)+d_n^*(x)$$

易见 $|d_n(1)|\le\int_{x_0}^{x_1}\frac{\varepsilon}{h}\,dx=\frac{\varepsilon}{h}\cdot h_n=\frac{\varepsilon}{n}$.

$|d_n^*(x)|\le\int_{x_1}^{x}\frac{\varepsilon}{h}\,dx\le\frac{\varepsilon}{h}\cdot h_n=\frac{\varepsilon}{n}$.

从而 $d_n(x)\triangleq d_n(1)+d_n^*(x)\le\frac{2\varepsilon}{n}$.

证明：对一般情形，即 $x\in(x_s,\ x_{s+1}]$ 时

$$d_n(i)\triangleq\int_{x_i}^{x_{i+1}}[f(x_i,y_i)-f(x,\varphi_n(x))]\,dx$$

$$d_n^*(x)\triangleq\int_{x_s}^{x}[f(x_s,y_s)-f(x,\varphi_n(x))]\,dx$$

$$\varphi_n(x) = y_s + \int_{x_s}^{x} f(x_s, y_s)\,dx$$

$$= y_s + \int_{x_s}^{x} f(x, \varphi_n(x))\,dx + d_n^*(x)$$

$$= y_{s-1} + \int_{x_{s-1}}^{x_s} f(x, \varphi_n(x))\,dx + d_{s-1}(x) + \int_{x_s}^{x} f(x, \varphi_n(x))\,dx + d_n^*(x)$$

$$= y_{s-1} + \int_{x_{s-1}}^{x} f(x, \varphi_n(x))\,dx + d_{s-1}(x) + d_n^*(x)$$

$$= \cdots = y_0 + \int_{x_0}^{x} f(x, \varphi_n(x))\,dx + \sum_{i=0}^{s-1} d_i(x) + d_n^*(x)$$

$$d_n(x) \triangleq \sum_{i=0}^{s-1} d_i(x) + d_n^*(x).$$

易见 $|d_i(x)| \le \int_{x_i}^{x_{i+1}} |f(x_i, y_i) - f(x, \varphi_n(x))|\,dx$

$$\le \int_{x_i}^{x_{i+1}} \frac{\varepsilon}{h}\,dx = \frac{\varepsilon}{n} \quad (n > N \text{ 时})$$

且由 ② 知 $|d_n^*(x)| \le \dfrac{\varepsilon}{n}$.

故 $|d_n(x)| \le \dfrac{s\varepsilon}{n} + \dfrac{\varepsilon}{n} \le \varepsilon$. $(n > N$ 时$)$

从而 $\lim\limits_{n\to\infty} d_n(x) = 0$ 原命题得证.

定理 3.3. 设 $f \in C(R)$ 则初值问题

(E): $\dfrac{dy}{dx} = f(x, y)$ $y(x_0) = y_0$ 在 $|x - x_0| \le h$ 上

至少有一个解 $y = y(x)$.

证明: 由引理 3.1. 可取子列 $\varphi_{n_1}(x) \cdots \varphi_{n_k}(x) \cdots$

s.t. $\varphi_{n_k}(x) \rightrightarrows \varphi(x)$ in $|x - x_0| \le h$.

由引理 3.2 知 $\varphi_{n_k}(x) = y_0 + \int_{x_0}^{x} f(x, \varphi_{n_k}(x))\,dx + d_{n_k}(x)$

$k \to \infty$ 得 $\varphi(x) = y_0 + \int_{x_0}^{x} f(x, \varphi(x))\,dx$

从而 $\varphi(x)$ 为 $|x - x_0| \le h$ 上的一个解.

例1：$\begin{cases} u_t = \Delta u + u^2 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ u|_{t=0} = u_0(x) > 0 \end{cases}$ 的解的存在性.

Step 1. 先考虑 $\begin{cases} u_t = u^2 & t > 0 \\ u|_{t=0} = u_0 \end{cases}$.

则 $\dfrac{du}{u^2} = dt \Rightarrow -\dfrac{1}{u} + \dfrac{1}{u_0} = t \Rightarrow u = \dfrac{u_0}{1 - t u_0}$.

若 $t_0 = \dfrac{1}{u_0}$，则有 $t \to t_0$ 时 $u \to +\infty$

Step 2 由 $\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \dfrac{\partial u}{\partial n} ds = 0$

则 $\int_\Omega u_t \, dx = \int_\Omega u^2 dx \geq \dfrac{1}{|\Omega|} \left(\int_\Omega u \, dx \right)^2$

定义 $v(t) = \int_\Omega u(x,t) \, dx$，则 $v_t \geq \dfrac{1}{|\Omega|} v^2$.

$v|_{t=0} = \int_\Omega u_0(x) \, dx > 0$.

设 $w(t)$ 满足 $\begin{cases} w_t = \dfrac{1}{|\Omega|} w^2 \\ w|_{t=0} = v(0) \end{cases}$.

再令 $z(t) \triangleq v(t) - w(t)$，则 z 满足

$\begin{cases} z'(t) \geq 0 \\ z|_{t=0} = 0 \end{cases}$ 故 $z(t) \geq 0 \Rightarrow v \geq w$.

由 Step 1 知 $\exists t_0$ s.t. $t \to t_0$ 时 $w(t) \to +\infty$

故 $v(t) \to +\infty$ ($t \to t_0$ 时). 若 u 整体存在.

则 $|u| \leq M$ in Ω. $v(t) = \int_\Omega u(x,t) \, dx$

$\leq M |\Omega| < +\infty$. 矛盾.

从而可知 $u(x,t)$ 在 Ω 中不是整体存在的

§3.3 解的延伸

定理3.4. 设 $P_0 \in G$，Γ 为 $\frac{dy}{dx}=f(x,y)$ 过 P_0 的积分曲线，则 Γ 在 G 内延伸到边界.

证明：设 $\Gamma: y=\varphi(x)$ $(x\in J)$. J 为最大存在区间

先考虑右侧延伸. 令 J^+ 为 Γ 在 P_0 右侧最大存在区间.

即 $J^+=J\cap[x_0,+\infty)$. 若结论不成立，则

(1) $J^+=[x_0,x_1]$ 有限 令 $y_1=\varphi(x_1)$，则 $(x_1,y_1)\in G$

$\because G$ 为开集 $\therefore \exists R_1: |x-x_1|\le a_1,\ |y-y_1|\le b_1$

s.t. $R_1\subset G$. 由定理3.3. 在 R_1 中有解

$y=\varphi_1(x)$. $|x-x_1|\le h_1$.

令 $y=\begin{cases}\varphi(x) & \text{当 } x_0\le x\le x_1\\ \varphi_1(x) & \text{当 } x_1\le x\le x_1+h_1\end{cases}$

则 $y\in C^1(G)$. 存在区间为 $[x_0,x_1+h_1]$. 矛盾.

(2) $J^+=[x_0,x_1)$ 半开区间. 先证 $\forall G_1\subset G$, G_1 有界闭

不可能有 $(x,\varphi(x))\in G_1$ $\forall x\in J^+$.

否则，设 G_1 为 G 的一个有限闭区域，使上式成立

则 $\varphi(x_0)=y_0$ $\varphi'(x)=f(x,\varphi(x))$

即 $\varphi(x)=y_0+\int_{x_0}^{x}f(x,\varphi(x))\,dx$ $(x_0\le x<x_1)$

定义 $K\triangleq\sup\limits_{G_1}|f(x,y)|$ 则 $|\varphi'(x)|\le K$ in J^+.

$|\varphi(x_1)-\varphi(x_2)|\le K|x_1-x_2|$. $x_1,x_2\in J^+$

若记 $y_1=\lim\limits_{x\to x_1}\varphi(x)$. 再定义

$\tilde{\varphi}(x)=\begin{cases}\varphi(x) & x_0\le x<x_1\\ y_1 & x=x_1\end{cases}$

易见 $\tilde{\varphi}(x)$ 在 $x_0 \le x \le x_1$ 上满足

$\tilde{\varphi}(x) = y_0 + \int_{x_0}^{x} f(x, \tilde{\varphi}(x))dx$. 即 $\tilde{\varphi}(x)$ 也是解

$\Rightarrow [x_0, x_1)$ 为最大存在区间矛盾.

综上, Γ 在 P_0 的右侧将延伸到 G 的边界. 左侧同理可证.

定理 3.5. $\frac{dy}{dx} = f(x, y)$ $f \in C(S)$.

$S: \alpha < x < \beta, \quad -\infty < y < +\infty$

且满足 $|f(x, y)| \le A(x)|y| + B(x)$

其中 $A(x) \ge 0$. $B(x) \ge 0$ 为连续函数. 则原方程

的每个解以 (α, β) 为最大存在区间.

证明: 先证右侧最大存在区间为 $[x_0, \beta)$

反之, 令右侧最大存在区间为 $[x_0, \beta_0)$ $(\beta_0 < \beta)$.

取 x_1, x_2 s.t. $x_0 < x_1 < \beta_0 < x_2 < \beta$

且 $x_2 - x_1 < x_1 - x_0$.

定义 $A_0 \triangleq \sup\limits_{[x_0, x_2]} A(x)$. $B_0 \triangleq \sup\limits_{[x_0, x_2]} B(x)$.

则 $|f(x, y)| \le A_0|y| + B_0$ $(x_0 \le x \le x_2, -\infty < y < \infty)$

不妨设 $a_1 \triangleq x_2 - x_1 < \frac{1}{4A_0}$

作 R_1: $|x - x_1| \le a_1$, $|y - y_1| \le b_1$

则 $|f(x, y)| \le A_0(|y_1| + b_1) + B_0$ $(x, y) \in R_1$

令 $M_1 \triangleq A_0(|y_1| + b_1) + B_0 + 1$. $h_1 = \min\left(a_1, \frac{b_1}{M_1}\right)$

作 R_1^*: $|x - x_1| \le h_1$, $|y - y_1| \le b_1$

由定理3.4. 过 (x_1, y_1) 的积分曲线 Γ 必可向右延伸到

R_1^* 边界. 且由 $|f(x, y)| \le M_1$ 知必停留在

$|y - y_1| \le M_1|x - x_1|$. $|x - x_1| \le h_1$

因此解可向右延伸至 $[x_0, x_1+h_1)$. 由于

$$a_1 < \frac{1}{4A_0} \quad 和 \quad \lim_{b_1\to\infty} \frac{b_1}{M_1} = \frac{1}{A_0} > \frac{1}{4A_0} > a_1$$

故 b_1 充分大时 $h_1 = a_1 = x_2 - x_1$. 从而解在 $[x_0, x_2)$ 上存在.

与假设矛盾 ($\because x_2 > \beta_0$). 从而结论成立.

同理可证左侧最大存在区间是 $(\alpha, x_0]$.

§3.4 三个极值原理:

回顾: Fermat 定理. 设 $u \in C^2(I)$ $x_0 \in I$

u 在 x_0 处取最大值. 则有 $u'(x_0) = 0$ 且 $u''(x_0) \le 0$

引理1: 设 $u(x) \in C^2(I)$ 其中 $I = (a, b)$

$u''(x) + g(x)u'(x) \ge 0$. 其中 $g \in C(I)$

则 $u(x)$ 在端点取最大值. 且如果在 I 内达到最大值

则必有 $u(x) \equiv C$.

证明:

Step 1: 若 $u''(x) + g(x)u'(x) > 0$ ...(*)

则若 $\exists x_0 \in I$. $u(x_0) = M = \sup\limits_{x\in[a,b]} u(x)$

有 $u''(x_0) \le 0$ 且 $u'(x_0) = 0$. 与(*)式矛盾.

Step 2: 令 $v(x) = u(x) + \varepsilon e^{\alpha x}$ (ε, α 待定)

则 $v' = u' + \varepsilon\alpha e^{\alpha x}$, $v'' = u'' + \varepsilon\alpha^2 e^{\alpha x}$

$$v'' + g(x)v' = u'' + g(x)u' + \varepsilon\alpha e^{\alpha x}(\alpha + g(x))$$

$$\ge \varepsilon\alpha e^{\alpha x}(\alpha + g(x)). \quad 取 \ \alpha = \max_{x\in[a,b]} |g(x)| + 1$$

则上式得 $v''(x) + g(x)v'(x) > 0$

由 Step 1 知 $\max\limits_{x\in[a,b]} v(x) \le \max\{v(a), v(b)\}$

故有 $\max\limits_{x\in[a,b]} u(x) \le \max\limits_{x\in[a,b]} v(x) \le \max\{v(a), v(b)\}$

$= \max\{u(a)+\varepsilon e^{\alpha a}, u(b)+\varepsilon e^{\alpha b}\}$

$\le \max\{u(a), u(b)\} + \varepsilon \max\{e^{\alpha a}, e^{\alpha b}\}$

令 $\varepsilon \to 0$ 得 $\max\limits_{x\in[a,b]} u(x) \le \max\{u(a), u(b)\}$.

Step 3: 若 $\exists\, c\in(a,b)$ s.t. $u(c)=M$

若 u 不恒为 M. 则 $\exists\, d\in(a,b)$ s.t. $u(d)<M$.

设 $c\in(a,d)$. 若令 $z(x)=e^{\alpha(x-c)}-1 \quad x\in[a,d]$

且 $v(x)=u(x)+\varepsilon z(x)$. 则有

$v'(x)=u'(x)+\varepsilon z'(x) = u'(x)+\alpha\varepsilon e^{\alpha(x-c)}$

$v''(x)=u''(x)+\alpha^2\varepsilon e^{\alpha(x-c)}$

$v''+gv'=u''+gu'+\alpha\varepsilon e^{\alpha(x-c)}\left(\alpha+g(x)\right)>0$

由 Step 1 知 v 在 a 或 d 处取最大值

而 $v(a)=u(a)+\varepsilon\left(e^{\alpha(a-c)}-1\right)<u(a)\le M$

$v(d)=u(d)+\varepsilon\left[e^{\alpha(d-c)}-1\right]<M$ （ε 充分小时）

而 $v(c)=M$. 故矛盾. 从而假设不成立.

从而 u 不能在 (a,b) 内取最大值.

以下推广到偏微分方程情形：

引理 2：若 $\Delta u+\sum\limits_{i=1}^{n} b_i u_i \ge 0$ in Ω

$u\in C^2(\Omega)\cap C(\bar{\Omega})$ 则 $\max\limits_{\bar{\Omega}} u(x) \le \max\limits_{\partial\Omega} u(x)$

证明：若 $\Delta u+\sum\limits_{i=1}^{n} b_i u_i > 0$. 且 $\exists\, x_0\in\Omega$

s.t. $u(x_0)=M=\max\limits_{\bar{\Omega}} u(x)$. 则必有 $Du(x_0)=0$

及 $D^2u(x_0)\le 0 \Rightarrow \Delta u\le 0$. 矛盾.

令 $v(x)=u(x)+\varepsilon e^{\alpha x_1}$. 则有

$v_1(x)=u_1(x)+\varepsilon\alpha e^{\alpha x_1}$. $\Delta v(x)=\Delta u(x)+\varepsilon\alpha^2 e^{\alpha x_1}$.

$$\Delta v+\sum_{i=1}^{n} b_i v_i=\Delta u+\sum_{i=1}^{n} b_i u_i+\varepsilon\alpha e^{\alpha x}\left(\alpha+b_1(x)\right)$$

令 $\alpha=\max\limits_{\bar\Omega}|b_1(x)|+1$. 则有 $\Delta v+\sum\limits_{i=1}^{n} b_i v_i>0$.

故由之前讨论知 $\max\limits_{\bar\Omega} v(x)\le\max\limits_{\partial\Omega} v(x)$.

从而 $\max\limits_{\bar\Omega} u(x)\le\max\limits_{\bar\Omega} v(x)\le\max\limits_{\partial\Omega}\left(u(x)+\varepsilon e^{\alpha x_1}\right)$

$\le\max\limits_{\partial\Omega} u(x)+\varepsilon e^{\alpha d}$. 其中 $d=\max\limits_{\bar\Omega}|x_1|$

令 $\varepsilon\to 0$ 知 $\max\limits_{\bar\Omega} u(x)\le\max\limits_{\partial\Omega} u(x)$.

引理3（Hopf 引理的一维情形）

$u(x)\in C^2(a,b)\cap C^1[a,b]$.

$u''(x)+g(x)u'(x)\ge 0$. $g(x)\in C[a,b]$

设 $u(a)=\max\limits_{x\in[a,b]} u(x)$. 则有 $u'(a)<0$

证明：取 $c\in(a,b)$ 则由引理1知 $u(c)<u(a)$

$v(x)\triangleq u(x)+\varepsilon\left(e^{\alpha(x-a)}-1\right)$ $\alpha\triangleq\max\limits_{x\in[a,b]}|g(x)|+1$

则 $v'(x)=u'(x)+\varepsilon\alpha e^{\alpha(x-a)}$

$v''(x)=u''(x)+\varepsilon\alpha^2 e^{\alpha(x-a)}$.

$$v''+gv'=u''+gu'+\varepsilon\alpha e^{\alpha(x-a)}\left(\alpha+g(x)\right)>0$$

故有 v 在 a 或 c 取最大值.

$v(a)=u(a)$ $v(c)=u(c)+\varepsilon\left(e^{\alpha(c-a)}-1\right)$

令 ε 充分小. 则有 $v(c)<u(a)=v(a)$.

从而 v 在 a 取最大值. 故有 $v'(a)\le 0$.

即 $u'(a)+\varepsilon\alpha\le 0$. 从而 $u'(a)\le-\varepsilon\alpha<0$.

这就证明了原定理.

注意: 定理中 $u'(a)\le 0$ 是平凡的. 主要是证明 $u'(a)<0$.

§3.5 比较定理

定理3.6 (第一比较定理)

设 $f(x,y)$, $F(x,y)\in C(G)$ 且 $f(x,y)<F(x,y)$

$y=\varphi(x)$ 与 $\Phi(x)$ 在 (a,b) 上分别为初值问题

(E_1): $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$

(E_2): $\frac{dy}{dx}=F(x,y)$, $y(x_0)=y_0$

的解. 其中 $(x_0,y_0)\in G$ 则我们有

$$\begin{cases}\varphi(x)<\Phi(x) & \text{当 } x_0<x<b \quad \cdots ① \\ \varphi(x)>\Phi(x) & \text{当 } a<x<x_0 \quad \cdots ②\end{cases}$$

证明: $\psi(x)\triangleq\Phi(x)-\varphi(x)$. 则 $\psi(x_0)=0$.

$\psi'(x_0)=F(x_0,y_0)-f(x_0,y_0)>0$

由定义知 $\exists\,\sigma>0$ s.t. $\psi(x)>0$ in $(x_0,x_0+\sigma)$

若 ① 不成立, 则有 $\exists\, x_1>x_0$ s.t. $\psi(x_1)=0$.

$\beta\triangleq\min\{x \mid \psi(x)=0,\ x\in(x_0,b)\}$

则有 $\begin{cases}\psi(\beta)=0\\ \psi(x)>0 \quad x\in(x_0,\beta)\end{cases}\Rightarrow \psi'(\beta)=\lim\limits_{x\to\beta^-}\frac{\psi(x)}{x-\beta}\le 0$

但 $\because \psi(\beta)=0$ $\therefore \gamma\triangleq\Phi(\beta)=\varphi(\beta)$

$\psi'(\beta)=F(\beta,\gamma)-f(\beta,\gamma)>0$. 矛盾.

从而 ① 成立. 同理可证 ② 也成立.

初值问题

(E): $\dfrac{dy}{dx}=f(x,y)$，$y(x_0)=y_0$，其中 $f\in C(R)$

R：$|x-x_0|\le a$，$|y-y_0|\le b$

$M\triangleq \max\limits_{(x,y)\in R}|f(x,y)|$　　$h\triangleq \min\left(a,\dfrac{b}{M}\right)$

若在 (x_0-h,x_0+h) 上 (E) 有两解 $y=Z(x)$ 和 $y=W(x)$

s.t. (E) 的任何解 $y=y(x)$ 满足

$W(x)\le y(x)\le Z(x)$　　$(|x-x_0|\le h)$

则称 $y=W(x)$ 和 $y=Z(x)$ 为最小解和最大解

由定义知其唯一性，以下证存在性。

定理3.7　$\exists\,\sigma<h$，在 $|x-x_0|\le\sigma$ 上，上述初值问题 (E) 有最小解和最大解

证明：考虑初值问题 (E_m)

(E_m): $\dfrac{dy}{dx}=f(x,y)+\varepsilon_m$　$y(x_0)=y_0$　其中 $\varepsilon_m\searrow 0$

由 Peano 定理知 $\exists\, h_m$ s.t. (E_m) 在 $|x-x_0|\le h_m$ 上有解

$y=\varphi_m(x)$　且 $h_m=\min\left\{a,\dfrac{b}{M+\varepsilon_m}\right\}$ 故 $h_m\to h$

$\therefore\ \exists\,\sigma_1<h$ s.t. m 充分大时 $h_m>\sigma_1$，$(m>N)$

$\sigma\triangleq\min\{h_1\cdots h_N,\sigma_1\}$　则有在 $I=(x_0-\sigma,x_0+\sigma)$

上 (E_m) 均有解

由 $\varphi_m(x)=y_0+\displaystyle\int_{x_0}^{x}[f(x,\varphi_m(x))+\varepsilon_m]\,dx$　$\cdots(*)$

知 $|\varphi_m(x)-y_0|\le(M+\varepsilon_m)h_m\le b$

即 $\{\varphi_m(x)\}$ 一致有界

又 $|\varphi_m(x_1)-\varphi_m(x_2)|\le\left|\displaystyle\int_{x_1}^{x_2}(f(x,\varphi_m(x))+\varepsilon_m)\,dx\right|$

$\le (M+\varepsilon_1)\,|x_1-x_2|$ 故 $\{\varphi_m(x)\}$ 等度连续

由 Arzela-Ascoli 引理知 ∃子列 $\varphi_{m_k}(x) \rightrightarrows \Phi(x)$

由(*)式知 $\Phi(x)$ 为(E)的一个解

下证其右行最大且左行最小，设 $y(x)$ 为原方程解

由第一比较定理得 $\begin{cases} y(x) < \varphi_{m_k}(x) & x\in(x_0,x_0+\sigma) \\ y(x) > \varphi_{m_k}(x) & x\in(x_0-\sigma,x_0) \end{cases}$

令 $k\to\infty$ 得 $\begin{cases} y(x) \le \Phi(x) & x\in(x_0,x_0+\sigma) \\ y(x) \ge \Phi(x) & x\in(x_0-\sigma,x_0) \end{cases}$

ε_m 换成 $-\varepsilon_m$ 可构造 $\varphi(x)$ 为左行最大 右行最小解

由在 (x_0,y_0) 处相切知可拼接出最大解与最小解

注：由解的延伸知可将最大解与最小解延伸到 G 的边界

定理 3.8 设 $f, F\in C(G)$，且 $f(x,y)\le F(x,y)$

$\varphi(x)$ 与 $\Phi(x)$ 分别为初值问题

(E_1): $\dfrac{dy}{dx}=f(x,y)$ $y(x_0)=y_0$

(E_2): $\dfrac{dy}{dx}=F(x,y)$ $y(x_0)=y_0$

的解，且 $\varphi(x)$ 为 (E_1) 右行最小 左行最大解

则有 $\begin{cases} \varphi(x) \le \Phi(x) & x_0\le x<b \quad \cdots ① \\ \varphi(x)\ge \Phi(x) & a<x\le x_0 \quad \cdots ② \end{cases}$

证明：定义初值问题 (E_m)

(E_m): $\dfrac{dy}{dx}=f(x,y)-\varepsilon_m$ $y(x_0)=y_0$

其中 $\varepsilon_m \searrow 0$，则由 $f(x,y)-\varepsilon_m < f(x,y)\le F(x,y)$

结合第一比较定理知，(E_m) 的解 $\varphi_m(x)$

满足 $\begin{cases} \varphi_m(x) < \Phi(x) & x_0<x<b \\ \varphi_m(x) > \Phi(x) & a<x<x_0 \end{cases}$

再结合定理3.7证明过程知$\{\varphi_m(x)\}$有子列

子列仍记为$\varphi_m(x) \rightrightarrows \varphi(x)$. 结合定理3.7的注

有 $\begin{cases} \varphi(x) \le \Phi(x) & x_0 \le x < b \\ \varphi(x) \ge \Phi(x) & a < x \le x_0 \end{cases}$.

同理可知. 若$\Phi(x)$为(E_2)的左行最小右行最大解.

则$\forall \varphi(x)$为(E_1)解. 有①. ②成立.

第四章 奇解与包络

§4.1 一阶隐式微分方程

例1. $y = xp + f(p)$ $\quad (p \triangleq \frac{dy}{dx})$

其中 $f''(p) \neq 0$.

解: $p = p + xp' + f'(p)\cdot p'$

即 $[x + f'(p)]\frac{dp}{dx} = 0$

当 $\frac{dp}{dx} = 0$ 时 $p = C$ $\quad y = Cx + f(C)$

当 $x + f'(p) = 0$ 时 得特解

$x = -f'(p)$ $\quad y = -f'(p)\cdot p + f(p)$

$\because f''(p) \neq 0$. $\quad x = -f'(p) \Rightarrow p = w(x)$.

代入得 $y = xw(x) + f(w(x))$ ---(*)

由 $y' = w(x) \Rightarrow$ (*) 的在 x_0 处切线为

$y = c_0 x + f(c_0)$ $\quad (c_0 = w(x_0))$

从而特解 (*) 在各点者随通解中某一解在该点与其相切.

例2: $x(y')^2 - 2yy' + 9x = 0$.

解: $y = \frac{9x}{2p} + \frac{xp}{2}$ $\quad (p = y')$

$\Rightarrow p = \frac{9}{2p} - \frac{9x}{2p^2}\cdot\frac{dp}{dx} + \frac{p}{2} + \frac{x}{2}\frac{dp}{dx}$

$\Rightarrow \left(\frac{1}{2} - \frac{9}{2p^2}\right)\left(p - x\frac{dp}{dx}\right) = 0$

$\Rightarrow \frac{dp}{dx} = \frac{p}{x}$ 或 $p^2 = 9$

$\Rightarrow \quad p = Cx$ 或 $p = \pm 3$.

通解 $y = \dfrac{9}{2C} + \dfrac{C}{2}x^2$. 特解 $y = \pm 3x$.

其主要特征为在这特解上每一点都有通解中的某个解在该点相切

例3（参数法）.

$$\left(\frac{dy}{dx}\right)^2 + y - x = 0.$$

令 $u = x$, $v = \dfrac{dy}{dx}$ 为两个参变量

由原方程 $\Rightarrow \quad x = u$, $\dfrac{dy}{dx} = v$, $y = u - v^2$.

$$\Rightarrow \quad v = \frac{du - 2v\,dv}{du} = 1 - 2v\frac{dv}{du}.$$

$$\Rightarrow \quad u = -2v - \ln(v-1)^2 + C \quad 或 \quad v \equiv 1.$$

$$\Rightarrow \begin{cases} x = C - 2v - \ln(v-1)^2 \\ y = C - 2v - \ln(v-1)^2 - v^2 \end{cases}$$

和特解 $\begin{cases} x = u \\ y = u - 1 \end{cases}$ （即 $y = x - 1$）.

例4： $(y-1)^2\left(\dfrac{dy}{dx}\right)^2 = \dfrac{4}{9}y$

解：设 $x = u$, $y = v$. 则 $\left(\dfrac{dv}{du}\right)^2 = \dfrac{4v}{9(v-1)^2}$

$p \triangleq \dfrac{dv}{du} = \dfrac{\pm 2\sqrt{v}}{3(v-1)}$ 分离变量得

$$\Rightarrow \quad \frac{3(v-1)}{2\sqrt{v}}\,dv = \pm du \Rightarrow \pm u = \sqrt{v}\,(v-3) + C$$

$\Rightarrow \quad (x - C)^2 = y(y-3)^2$ 为通解.

特解为 $y \equiv 0$ （$\because y \equiv 3$ 代入不满足原方程）

§4.2 奇解

定义4.1. $F(x,y,\frac{dy}{dx})=0$ 为一阶微分方程.

其有特解 $\Gamma: y=\varphi(x)$. $(x\in J)$

若 $\forall Q\in\Gamma$. 在 Q 的任何邻域内方程有不同于 Γ 的解

在 Q 点与 Γ 相切. 则称 Γ 为奇解.

例2中的 $y=\pm 3x$ 即为原方程的奇解.

定理4.1. $F(x,y,p)\in C(G)$. 且 F_y'. F_p' 连续

若 $y=\varphi(x)$. $(x\in J)$ 为一个奇解. 并且有

$(x,\ \varphi(x),\ \varphi'(x))\in G\quad (x\in J)$

则 $\varphi(x)$ 满足 p-判别式:

$$F(x,y,p)=0.\quad F_p'(x,y,p)=0.$$

证明: $\because\ \varphi(x)$ 为 $F(x,y,p)=0$ 的解显然有第一式

反设其不满足第二式. $\exists x_0\in J$. s.t. $F_p(x_0,y_0,p_0)\neq 0$.

由隐函数定理知原方程 $F(x,y,p)=0$ 在 (x_0,y_0)

附近唯一地确定了 $\frac{dy}{dx}=f(x,y)$ --- $(*)$

其中 $f(x_0,y_0)=p_0$. 从而 $F(x,y,p)=0$ 的解

必为 $\frac{dy}{dx}=f(x,y)$ 的解.

另一方面. 由于 $f(x,y)$ 在 (x_0,y_0) 某邻域内连续

$$f_y'(x,y)=-\frac{F_y'(x,y,f(x,y))}{F_p'(x,y,f(x,y))}\quad \text{连续}$$

由Picard定理知 $(*)$ 满足 $y(x_0)=y_0$ 的解存在唯一.

从而 $\varphi(x)$ 为 x_0 处局部唯一解. 从而 x_0 附近不存在

其它解与之相切. 与奇解定义矛盾.

例2： $x(y')^2-2yy'+9x=0$.

由之前讨论知 通解为 $y=\frac{C}{2}x^2+\frac{9}{2C}$.

特解 $y=\pm 3x$. 代入P判别式有 $P=\pm 3$.

$\Rightarrow y-px=0 \Rightarrow F_p'=2px-2y=0$

$\Rightarrow$ 特解 $y=\pm 3x$ 满足P判别式.

§4.3. 包络

定义4.2. Γ 为连续可微曲线 若 $\forall q\in\Gamma$. 在曲线簇 $K(C)$：$V(x,y,C)=0$ 中都有一条曲线 $K(C^*)$ 过 q 并在该点与 Γ 相切. 且 $K(C^*)$ 在 q 某一邻域内不同于 Γ. 则称 Γ 为 $K(C)$ 的一支包络.

定理4.3. $F(x,y,\frac{dy}{dx})=0$ 有通积分 $U(x,y,C)=0$. 又设其有包络 Γ：$y=\varphi(x)$. 则 $y=\varphi(x)$ 为 $F(x,y,\frac{dy}{dx})=0$ 的奇解.

证明：由定义知 只要证 $\varphi(x)$ 是解即可.

取 $(x_0,y_0)\in\Gamma$. 由包络定义知 $U(x,y,C)=0$ 中有 $y=u(x,C_0)$ 在 (x_0,y_0) 与 Γ 相切.

即 $\varphi(x_0)=u(x_0,C_0)$ $\quad \varphi'(x_0)=u_x'(x_0,C_0)$

$\because y=u(x,C_0)$ 为原方程解.

$\therefore F(x_0,u(x_0,C_0),u_x'(x_0,C_0))=0$

因此 $F(x_0,\varphi(x_0),\varphi'(x_0))=0$

由 x_0 任意性知 $\varphi(x)$ 为原方程解

定理4.4. 设 Γ 为 $V(x,y,C)=0$ 的一支包络. 则它满足 C判别式 $V(x,y,C)=0$. $V_C'(x,y,C)=0$.

先看例2　$x(y')^2-2yy'+9x=0$

由之前讨论知　$y=\frac{c}{2}x^2+\frac{9}{2c}$ 为通解

且 $y=\pm 3x$ 为两个特解

$$V(x,y,c)=\frac{c}{2}x^2-y+\frac{9}{2c}$$

$$V_c'(x,y,c)=\frac{x^2}{2}-\frac{9}{2c^2}$$

故 $\begin{cases} V(x,y,c)=0 \\ V_c'(x,y,c)=0 \end{cases} \Longleftrightarrow \begin{cases} y=\frac{c}{2}x^2+\frac{9}{2c} \\ c^2x^2=9 \end{cases}$

消去 C 得 $y=\pm 3x$　故特解满足 C-判别式.

定理4.4的证明：

由包络定义可设 $x=f(c),\ y=g(c)\quad (c\in I)$

则有 $V(f(c),g(c),c)=0,\ \forall c\in I$

设 $f,g\in C^1(I)$，则 $V_x'f'(c)+V_y'g'(c)+V_c'=0$

$\forall c\in I$，当 $(f'(c),g'(c))=0$ 或 $(V_x',V_y')=(0,0)$ 时

有 $V_c'(f(c),g(c),c)=0$.

当 $(f'(c),g'(c))\neq(0,0)$ 和 $(V_x',V_y')\neq(0,0)$ 时

此时 Γ 在 $q(c)=(f(c),g(c))$ 的切向量 $(f'(c),g'(c))$

和在 $q(c)$ 的 $V(x,y,c)=0$ 的切向量 $(-V_y',V_x')$ 非退化.

从而 $(f'(c),g'(c))\,//\,(-V_y',V_x')$

即 $f'(c)V_x'+g'(c)V_y'=0$，从而 $V_c'=0$.

于是 $\forall c\in I$，$V(f(c),g(c),c)=0\Rightarrow V_c'=0$.

这就证明了 C-判别式.

第五章　高阶微分方程

§5-1. 万有引力与开普勒三大定律

开普勒三大定律

第一定律：所有行星围绕太阳运动的轨道是椭圆

第二定律：每一个行星在绕太阳运动时面积速度不变

第三定律：所有行星的轨道半长轴 a 与运动周期 T

满足 $\dfrac{a^3}{T^2} \equiv C$，$C$ 只与太阳质量有关.

证明. step 1：先证第二定律以及行星运动为平面运动

记太阳质量 M，行星质量 m，引力常数为 G.

且令 $k \triangleq GM$，从而由牛顿第二定律与万有引力定律

$$-\frac{GMm}{r^2}\cdot\frac{\vec{r}}{r} = m\frac{d^2\vec{r}}{dt^2}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -\frac{k}{r^2}\cdot\frac{\vec{r}}{r} \quad \cdots (*)$$

角动量 $\vec{L}(t) = m\dot{\vec{r}}(t) \times \vec{r}(t)$

$$\frac{d\vec{L}}{dt} = m\ddot{\vec{r}} \times \vec{r} + m\dot{\vec{r}} \times \dot{\vec{r}}$$

$$= m\cdot\left(-\frac{k\cdot\vec{r}}{r^3}\right)\times\vec{r} = 0.$$

故 $\vec{L}(t) \equiv \vec{C}$. $\because \vec{r}(t) \perp \vec{L}(t)$.

而 $\vec{L}(t)$ 为常向量. $\therefore \vec{r}(t)$ 在一个平面内.

从而可设 $\vec{r}(t) = (x(t), y(t))$

$$\vec{r} = r\cos\theta\,\vec{e_x} + r\sin\theta\,\vec{e_y}$$

$$\frac{d\vec{r}}{dt} = (\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta})\vec{e_x} + (\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta})\vec{e_y}$$

又 $\vec{L}(t) = m\dot{\vec{r}}(t) \times \vec{r}(t) = m[(\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta})\vec{e_x}$
$+(\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta})\vec{e_y}] \times (r\cos\theta\,\vec{e_x} + r\sin\theta\,\vec{e_y})$
$= m(-r^2\sin^2\theta\cdot\dot{\theta} - r^2\cos^2\theta\cdot\dot{\theta})\vec{e_z} = -mr^2\dot{\theta}\,\vec{e_z}$.

由 $\vec{L}(t) \equiv \vec{C}$ 知 $r^2\frac{d\theta}{dt} \equiv L$ 为常数.

另一方面，$\Delta S = S(t+\Delta t) - S(t)$
$= \frac{1}{2}r^2(t)\frac{d\theta}{dt}\cdot\Delta t + o(\Delta t)$
$\Rightarrow \frac{dS}{dt} = \frac{1}{2}r^2(t)\frac{d\theta}{dt} \equiv C$. 即面积速度不变

Step 2：证明轨道为椭圆，太阳位于一个焦点上.

由 $r^2\dot{\theta} \equiv C \Rightarrow 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$. $\because r>0$.

$\therefore 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$.

从而有 $\frac{d^2\vec{r}}{dt^2} = (\ddot{r}\cos\theta - 2\dot{r}\sin\theta\cdot\dot{\theta} - r\cos\theta\cdot\dot{\theta}^2 - r\sin\theta\cdot\ddot{\theta})\vec{e_x}$
$+(\ddot{r}\sin\theta + 2\dot{r}\cos\theta\cdot\dot{\theta} - r\sin\theta\,\dot{\theta}^2 + r\cos\theta\cdot\ddot{\theta})\vec{e_y}$
$= (\ddot{r}\cos\theta - r\cos\theta\cdot\dot{\theta}^2)\vec{e_x} + (\ddot{r}\sin\theta - r\sin\theta\cdot\dot{\theta}^2)\vec{e_y}$
$= (\ddot{r} - r\dot{\theta}^2)(\cos\theta\,\vec{e_x} + \sin\theta\,\vec{e_y}) = (\ddot{r} - r\dot{\theta}^2)\frac{\vec{r}}{r}$.

又由 (*) 式 $\frac{d^2\vec{r}}{dt^2} = -\frac{k}{r^2}\cdot\frac{\vec{r}}{r}$

从而有 $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \frac{k}{r^2} = 0$.

而 $L = r^2\frac{d\theta}{dt}$. 故上式可化为 $\frac{d^2r}{dt^2} = \frac{L^2}{r^3} - \frac{k}{r^2}$.

能量：$E(t) = \frac{1}{2}mv^2 - \frac{GMm}{r}$.

由 $\vec{v}=\frac{d\vec{r}}{dt}$ 知 $\quad \frac{1}{2}mv^2=\frac{1}{2}m\left|\frac{d\vec{r}}{dt}\right|^2$

$$=\frac{m}{2}\left[(\dot{r}\cos\theta-r\sin\theta\cdot\dot{\theta})^2+(\dot{r}\sin\theta+r\cos\theta\cdot\dot{\theta})^2\right]$$

$$=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)=\frac{1}{2}m\left(\dot{r}^2+\frac{L^2}{r^2}\right)$$

$$E(t)=\frac{1}{2}m\left(\frac{dr}{dt}\right)^2+\frac{m}{2}\frac{L^2}{r^2}-\frac{km}{r}$$

$$\frac{dE}{dt}=m\frac{dr}{dt}\cdot\frac{d^2r}{dt^2}-\left(\frac{mL^2}{r^3}+\frac{km}{r^2}\right)\cdot\frac{dr}{dt}$$

$$=m\frac{dr}{dt}\left(\frac{d^2r}{dt^2}-\frac{L^2}{r^3}+\frac{k}{r^2}\right)=0.$$

故能量守恒. $E(t)\equiv E$.

即 $\frac{1}{2}\left(\frac{dr}{dt}\right)^2+\frac{L^2}{2r^2}-\frac{k}{r}=\frac{E}{m}$.

记 $V=m\left(\frac{L^2}{2r^2}-\frac{k}{r}\right)$. 则有

$$\frac{dr}{dt}=\sqrt{\frac{2(E-V)}{m}} \quad \text{又有} \quad \frac{d\theta}{dt}=\frac{L}{r^2}.$$

从而 $\quad \frac{d\theta}{dr}=\frac{L}{r^2}\sqrt{\frac{m}{2(E-V)}}$

$$\theta=\int\frac{L}{r^2}\sqrt{\frac{m}{2(E-V)}}\,dr$$

$$\overset{\rho\triangleq\frac{1}{r}}{=\!=}-L\sqrt{\frac{m}{2}}\int\frac{d\rho}{\sqrt{E-V}}=-L\sqrt{\frac{m}{2}}\int\frac{d\rho}{\sqrt{E+\left(k\rho-\frac{L^2\rho^2}{2}\right)m}}$$

$$=-L\sqrt{\frac{1}{2}}\int\frac{\frac{1}{L}\,d\left(L\rho-\frac{k}{L}\right)}{\frac{1}{\sqrt{2}}\sqrt{-\left(L\rho-\frac{k}{L}\right)^2+\frac{2E}{m}+\frac{k^2}{L^2}}}$$

$$= \int \frac{d\left(L\rho - \frac{k}{L}\right)}{\sqrt{\frac{2E}{m} + \frac{k^2}{L^2} - \left(L\rho - \frac{k}{L}\right)^2}}$$

$$= \arccos\left(\frac{\frac{L}{r} - \frac{k}{L}}{\sqrt{\frac{2E}{m} + \frac{k^2}{L^2}}} \right)$$

从而 $\cos\theta = \dfrac{\frac{L}{r} - \frac{k}{L}}{\sqrt{\frac{2E}{m} + \frac{k^2}{L^2}}} \Rightarrow r = \dfrac{p}{1 + e\cos\theta}$.

其中 $p = \dfrac{L^2}{k}$. $e = \sqrt{1 + \dfrac{2EL^2}{mk^2}}$.

当 $E < 0$ 时 易见 $e < 1$. 从而行星为椭圆轨道.

Step 3: 开普勒第三定律

$\theta \in [0, 2\pi]$. $t \in [0, T]$.

由 $\dfrac{dS}{dt} = \dfrac{1}{2} r^2 \dfrac{d\theta}{dt} = \dfrac{1}{2} L$.

从而 $\displaystyle\int_0^T \frac{dS}{dt} \cdot dt = \frac{1}{2} L T = \pi a b$.

$\Rightarrow T = \dfrac{2\pi a b}{L}$. 而 $a = \dfrac{p}{1 - e^2} = -\dfrac{mk}{2E}$

$b = \sqrt{a^2 - c^2} = a\sqrt{1 - e^2} = L\sqrt{\dfrac{m}{2|E|}}$

从而 $T = \dfrac{2\pi \cdot \frac{mk}{2|E|} \cdot L\sqrt{\frac{m}{2|E|}}}{L} = \dfrac{\pi k}{\sqrt{2}} \left(\dfrac{m}{|E|}\right)^{\frac{3}{2}}$.

$= \dfrac{\pi k}{\sqrt{2}} \left(\dfrac{2a}{k}\right)^{\frac{3}{2}} = \left(\dfrac{4\pi^2}{k} \cdot a^3\right)^{\frac{1}{2}} \Rightarrow \dfrac{a^3}{T^2} = \dfrac{k}{4\pi^2}$

从而第三定律得证.

§5.2 解对初值和参数的连续依赖性

例 1：
$$\begin{cases} \dfrac{d^2x}{dt^2} + a^2x = 0 . \\ x(t_0) = x_0 , \quad x'(t_0) = v_0 . \end{cases}$$

若令 $\begin{cases} y_1 = x \\ y_2 = \dfrac{dx}{dt} \end{cases}$ 则有 $\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -a^2 y_1 \end{cases}$ 且 $\begin{cases} y_1(t_0) = x_0 \\ y_2(t_0) = v_0 \end{cases}$

易见其解为 $x = x_0 \cos a(t-t_0) + \dfrac{v_0}{a} \sin a(t-t_0)$

知其对 x_0，v_0，t_0 均连续可微．（$a \neq 0$ 时对 a 连续可微）

例 2：
$$\begin{cases} \Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \quad \Omega \text{ bdd} \\ u = \varphi & \text{on } \partial\Omega . \end{cases}$$

① 考虑 $\begin{cases} \Delta u_1 = 0 & \text{in } \Omega \\ u_1 = \varphi_1 & \text{on } \partial\Omega \end{cases}$ 与 $\begin{cases} \Delta u_2 = 0 & \text{in } \Omega \\ u_2 = \varphi_2 & \text{on } \partial\Omega . \end{cases}$

则易见 $\begin{cases} \Delta(u_1 - u_2) = 0 & \text{in } \Omega \\ u_1 - u_2 = \varphi_1 - \varphi_2 & \text{on } \partial\Omega \end{cases}$

由极值原理知 $\max\limits_{\Omega} |u_1 - u_2| \leq \max\limits_{\partial\Omega} |\varphi_1 - \varphi_2|$

从而解关于 φ 连续依赖．

② 考察 $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

$v \triangleq \dfrac{|x|^2 - d^2}{2n} \max\limits_{\Omega} |f|$． 其中 $d = \operatorname{diam}(\Omega)$．

则有 $\begin{cases} \Delta v = \max\limits_{\Omega} |f| & \text{in } \Omega \\ v \leq 0 & \text{on } \partial\Omega . \end{cases}$

$$\begin{cases} \Delta(u+v) = \Delta u + \Delta v \geq 0 & \text{in } \Omega \\ u+v \leq 0 & \text{on } \partial\Omega \end{cases}$$

由极值原理知 $u+v \le 0$ in Ω.

同理 $\begin{cases} \Delta(u-v) \le 0 & \text{in } \Omega \\ u-v \ge 0 & \text{on } \partial\Omega \end{cases} \Rightarrow u \ge v$ in Ω.

故有 $\dfrac{|x|^2-d^2}{2n} \max\limits_{\Omega} |f| \le u \le \dfrac{d^2-|x|^2}{2n} \max\limits_{\Omega} |f|$

$\Rightarrow |u| \le \dfrac{d^2}{2n} \max\limits_{\Omega} |f|$.

从而. 若 $\begin{cases} \Delta u_1 = f_1 & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega \end{cases}$ 且 $\begin{cases} \Delta u_2 = f_2 & \text{in } \Omega \\ u_2 = 0 & \text{on } \partial\Omega \end{cases}$

则有 $\begin{cases} \Delta(u_1-u_2) = f_1 - f_2 & \text{in } \Omega \\ u_1 - u_2 = 0 & \text{on } \partial\Omega \end{cases}$

$\Rightarrow \max\limits_{\Omega} |u_1-u_2| \le \dfrac{d^2}{2n} \max\limits_{\Omega} |f_1-f_2|$

从而解关于 f 也连续依赖.

Gronwall 不等式: 设 $g(t), u(t) \in C[t_0, t_1]$ 且非负

$C, K \ge 0$ 为常数 若 $\forall t \in [t_0, t_1]$ 有

$u(t) \le C + \int_{t_0}^{t} [g(s)u(s)+K]\,ds$. 则有

$u(t) \le [C + K(t-t_0)]\, e^{\int_{t_0}^{t} g(s)\,ds}$.

证明: ① 当 $C>0$ 时

由 $\dfrac{g(t)u(t)+K}{C+\int_{t_0}^{t}[g(s)u(s)+K]ds} \le g(t) + \dfrac{K}{C+\int_{t_0}^{t}[g(s)u(s)+K]ds}$

$\le g(t) + \dfrac{K}{C+K(t-t_0)}$ 两边从 t_0 到 t 积分有

$\ln\left(C+\int_{t_0}^{t}[g(s)u(s)+K]ds\right) - \ln C \le \int_{t_0}^{t} g(s)\,ds$

$+ \ln(C+K(t-t_0)) - \ln C$

两边取 exp 有

$$C+\int_{t_0}^{t}[g(s)u(s)+K]\,ds \le [C+K(t-t_0)]e^{\int_{t_0}^{t}g(s)\,ds}$$

从而有 $u(t) \le [C+K(t-t_0)]e^{\int_{t_0}^{t}g(s)\,ds}$.

② $C=0$ 时，在①中令 $C\to 0^+$ 即得结论 .

引理：方程 $\begin{cases}\dfrac{d\vec{x}}{dt}=f(t,\vec{x})\\ \vec{x}(t_0)=\vec{y_0} \text{ 或 } \vec{x}(t_0)=\vec{z_0}\end{cases}$ 的解在 $[t_0,t_1]$ 上存在 且 f 局部 Lipschitz 连续

则 $\forall t\in[t_0,t_1]$ 有

$$\|x(t,y_0)-x(t,z_0)\| \le \|y_0-z_0\|e^{L(t-t_0)}$$

注：f 局部 Lipschitz 连续是指 $\forall x_0\in W\subseteq \mathbb{R}^n$ ，$\exists B_{\varepsilon_0}(x_0)$

s.t. $f(x)$ 在 $B_{\varepsilon_0}(x_0)$ 上满足 Lipschitz 条件 .

证明：$B\triangleq\{x \mid x=x(t,y_0) \text{ 或 } x=x(t,z_0),\ t\in[t_0,t_1]\}$

则 $\exists A$ 有界闭，$B\subset A\subset W$ ，由有限覆盖定理知 .

$f(t,x)$ 在 A 上有一致的 Lipschitz 常数 L .

由 $x(t,y_0)=y_0+\int_{t_0}^{t}f(s,x(s,y_0))\,ds$

$x(t,z_0)=z_0+\int_{t_0}^{t}f(s,x(s,z_0))\,ds$

$$\Rightarrow \|x(t,y_0)-x(t,z_0)\| \le \|y_0-z_0\|$$

$$+\int_{t_0}^{t}\|f(s,x(s,y_0))-f(s,x(s,z_0))\|\,ds$$

$$\le \|y_0-z_0\|+L\int_{t_0}^{t}\|x(s,y_0)-x(s,z_0)\|\,ds$$

定义 $u(t)\triangleq\|x(t,y_0)-x(t,z_0)\|$ 则 $u(t_0)=\|y_0-z_0\|$

$$u(t)\le u(t_0)+L\int_{t_0}^{t}u(s)\,ds$$

由 Gronwall 不等式知 $u(t)\le u(t_0)e^{L(t-t_0)}$

此即 $\|x(t,y_0)-x(t,z_0)\| \le \|y_0-z_0\|e^{L(t-t_0)}$

定理 5.2 若方程 $\begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \\ \vec{x}(t_0) = y_0 \end{cases}$

的解在 $[t_0, t_1]$ 上有定义，则 $\exists$ y_0 的邻域 $V \subset W$，

s.t. $z_0 \in V$ 时，方程关于初值 z_0 的解 $x(t, z_0)$ 在

$[t_0, t_1]$ 上有定义，且当 $t \in [t_0, t_1]$ 时

$$\|x(t, y_0) - x(t, z_0)\| \leq \|y_0 - z_0\| e^{L(t-t_0)}$$

证明：$A \triangleq \{x \mid x \in W, \|x - x(t, y_0)\| \leq \varepsilon, t \in [t_0, t_1]\}$

则 A 有界闭，$A \subset W$，取 $\delta > 0$ 使 $\delta e^{L(t_1 - t_0)} < \frac{\varepsilon}{2}$

令 $\|z_0 - y_0\| < \delta$，要证 $x(t, z_0)$ 在 $[t_0, t_1]$ 上有定义

反设其最大存在区间为 $[t_0, \beta)$，且 $\beta \leq t_1$。

由定理 3.4 知 $x(t, z_0)$ 会延伸到 A 外，故

$\exists t' \in [t_0, \beta)$ s.t. $x(t', z_0) \in \partial A$，

且当 $t \in [t_0, t')$ 时 $x(t, z_0) \in A$。

一方面，由引理知

$$\|x(t', y_0) - x(t', z_0)\| \leq \|y_0 - z_0\| e^{L(t'-t_0)}$$

$$\leq \delta e^{L(t_1 - t_0)} < \frac{\varepsilon}{2}$$

另一方面 $\|x(t', y_0) - x(t', z_0)\| \geq \varepsilon$，矛盾

故有 $\beta > t_1$，即解在 $[t_0, t_1]$ 上有定义。

结合引理知本定理得证。

注：以上 $\|\cdot\|$ 为欧氏空间范数

即 $\|(x_1 \cdots x_n)\| \triangleq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

由柯西不等式易知 $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

§5.3 解对初值和参数的连续可微性

只考虑 $\begin{cases} \dfrac{d\vec{y}}{dx} = \vec{f}(x, \vec{y}, \vec{\lambda}) \\ \vec{y}(0) = \vec{0} \end{cases}$ 的解 $\vec{y} = \vec{\varphi}(x, \lambda)$ 对 λ 的连续可微性

例 1：$\begin{cases} \dfrac{dy}{dx} = f(x, y, \lambda) \\ y(x_0) = y_0 \end{cases}$

计算 $\dfrac{\partial y}{\partial x_0}$, $\dfrac{\partial y}{\partial y_0}$, $\dfrac{\partial y}{\partial \lambda}$

解：考虑与之等价的积分方程

$$\varphi(x; x_0, y_0, \lambda) = y_0 + \int_{x_0}^{x} f(x, \varphi(x; x_0, y_0, \lambda), \lambda)\, dx$$

则
$$\frac{\partial \varphi}{\partial x_0} = -f(x_0, y_0) + \int_{x_0}^{x} \frac{\partial f}{\partial y} \frac{\partial \varphi}{\partial x_0}\, dx$$

$$\frac{\partial \varphi}{\partial y_0} = 1 + \int_{x_0}^{x} \frac{\partial f}{\partial y} \frac{\partial \varphi}{\partial y_0}\, dx$$

$$\frac{\partial \varphi}{\partial \lambda} = \int_{x_0}^{x} \left(\frac{\partial f}{\partial y} \frac{\partial \varphi}{\partial \lambda} + \frac{\partial f}{\partial \lambda} \right) dx$$

令 $z = \dfrac{\partial \varphi}{\partial \lambda}$ $\quad z$ 满足初值问题

$$\begin{cases} \dfrac{dz}{dx} = \dfrac{\partial f}{\partial y} \cdot z + \dfrac{\partial f}{\partial \lambda} \\ z(x_0) = 0 \end{cases}$$

例 2：$\begin{cases} \dfrac{dy}{dx} = \sin(\lambda x y) \\ y(x_0) = y_0 \end{cases}$ 求 $\left. \dfrac{\partial \varphi}{\partial \lambda} \right|_{(x_0, y_0) = (0, 0)}$

解：在例 1 中令 $f(x, y, \lambda) = \sin(\lambda x y)$

$z \triangleq \dfrac{\partial \varphi}{\partial \lambda}$ 则有 $\begin{cases} \dfrac{dz}{dx} = \lambda x \cos(\lambda x y) z + x y \cos(\lambda x y) \\ z(x_0) = 0 \end{cases}$

从而若令 $(x_0, y_0) = (0, 0)$，代入有 $z(0) = 0$。

此即 $\left.\dfrac{\partial\varphi}{\partial\lambda}\right|_{(x_0, y_0)=(0,0)} = 0$。

下面讨论参数 λ 为一维时的解对参数连续可微性定理

定理 5.3 $\begin{cases} \dfrac{d\vec{y}}{dx} = \vec{f}(x, \vec{y}, \lambda) \\ \vec{y}(0) = \vec{0} \end{cases}$。

且设 $\vec{f} \in C(G)$ $G: |x| \le a$ $|\vec{y}| \le b$ $|\lambda - \lambda_0| \le \rho$。

且 $\vec{f}$ 对 $\vec{y}, \lambda$ 有连续偏导数

则方程解 $\vec{y} = \vec{\varphi}(x, \lambda)$ 在 $D: |x - x_0| \le h, |\lambda - \lambda_0| \le \rho$

上是连续可微的。

证明：由 $\vec{y}(x, \lambda_0) = y_0 + \int_{x_0}^{x} f(s, y(s, \lambda_0), \lambda_0)\, ds$

知 $\dfrac{\Delta\vec{y}}{\Delta\lambda} = \dfrac{\vec{y}(x, \lambda_0 + \Delta\lambda) - \vec{y}(x, \lambda_0)}{\Delta\lambda}$

$$= \int_{x_0}^{x} \frac{\vec{f}(s, y(s, \lambda_0 + \Delta\lambda), \lambda_0 + \Delta\lambda) - \vec{f}(s, y(s, \lambda_0), \lambda_0)}{\Delta\lambda}\, ds$$

$$= \int_{x_0}^{x} \left[\frac{\partial\vec{f}}{\partial\vec{y}}(s, y(s, \lambda_0), \lambda_0) + \varepsilon_1\right] \frac{\Delta y}{\Delta\lambda}\, ds$$

$$+ \int_{x_0}^{x} \left(\frac{\partial\vec{f}}{\partial\lambda}(s, y(s, \lambda_0), \lambda_0) + \vec{\varepsilon_2}\right) ds$$

其中 ε_1 为 $n\times n$ 方阵，$\vec{\varepsilon_2}$ 为 n 维向量 $\varepsilon_1, \vec{\varepsilon_2} \to 0$ $(\Delta\lambda \to 0)$

令 $\vec{z} = \dfrac{\partial\vec{y}}{\partial\lambda}$ 由例1知 $\dfrac{d\vec{z}}{dx} = \dfrac{\partial\vec{f}}{\partial\vec{y}} z + \dfrac{\partial\vec{f}}{\partial\lambda}$

故 $\vec{z} = \int_{x_0}^{x} \dfrac{\partial\vec{f}}{\partial\vec{y}}(s, y(s, \lambda_0), \lambda_0)\, z + \dfrac{\partial\vec{f}}{\partial\lambda}(s, y(s, \lambda_0), \lambda_0)\, ds$

从而 $\dfrac{\Delta\vec{y}}{\Delta\lambda} - \vec{z} = \int_{x_0}^{x} \dfrac{\partial\vec{f}}{\partial\vec{y}}\left(\dfrac{\Delta y}{\Delta\lambda} - z\right) + \varepsilon_1 \dfrac{\Delta y}{\Delta\lambda} + \vec{\varepsilon_2}\, ds$

故有 $\left\|\frac{\Delta\vec{y}}{\Delta\lambda}-\vec{z}\right\| \leq \int_{x_0}^{x}\left\|\frac{\partial\vec{f}}{\partial\vec{y}}(s,\vec{y}(s,\lambda_0),\lambda_0)\left(\frac{\Delta\vec{y}}{\Delta\lambda}-\vec{z}\right)\right\|ds$

$+\int_{x_0}^{x}\left\|\varepsilon_1\left(\frac{\Delta\vec{y}}{\Delta\lambda}-\vec{z}\right)+\varepsilon_1\vec{z}+\vec{\varepsilon_2}\right\|ds$

由 f 关于 $\vec{y}$、λ 有连续偏导数知 $\exists M>0$.

s.t. $\left\|\frac{\partial\vec{f}}{\partial\vec{y}}\right\|\leq M$. 且 $\|\vec{z}\|\leq M$.

$K\triangleq\varepsilon_1\vec{z}+\vec{\varepsilon_2}\to 0\quad(\Delta\lambda\to 0)$. 从而有

$$\left\|\frac{\Delta\vec{y}}{\Delta\lambda}-\vec{z}\right\|\leq\int_{x_0}^{x}\left[M\left\|\frac{\Delta\vec{y}}{\Delta\lambda}-\vec{z}\right\|+K\right]ds$$

由 Gronwall 不等式知

$$\left\|\frac{\Delta\vec{y}}{\Delta\lambda}-\vec{z}\right\|\leq K(x-x_0)e^{M(x-x_0)}$$

由 $\lim\limits_{\Delta\lambda\to 0}K=0$ 知 $\lim\limits_{\Delta\lambda\to 0}\left\|\frac{\Delta\vec{y}}{\Delta\lambda}-\vec{z}\right\|=0$.

故 $\frac{\Delta\vec{y}}{\Delta\lambda}\to\vec{z}\quad(\Delta\lambda\to 0)$.

从而 $\vec{y}$ 关于 λ 连续可微. 定理得证

第六章 线性微分方程组

§6.1. 一般理论

$$\frac{dy_i}{dx} = \sum_{j=1}^{n} a_{ij}(x)\, y_j + f_i(x). \quad (i=1,2\cdots n)$$

a_{ij}. f_i 在 $(a.b)$ 上连续

可记为 $\dfrac{d\vec{y}}{dx} = A\vec{y} + \vec{f}$ (*) $\vec{f}\equiv\vec{0}$ 时称为齐次的.

例1：验证方程组

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos^2 x & \frac{1}{2}\sin 2x - 1 \\ \frac{1}{2}\sin 2x + 1 & \sin^2 x \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

的通解为 $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1\begin{pmatrix} e^x\cos x \\ e^x \sin x \end{pmatrix} + C_2\begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}$

证明：事实上只要证 $\begin{pmatrix} e^x\cos x \\ e^x\sin x \end{pmatrix}$ 与 $\begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}$ 为解即可

例如 $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}$ 时

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\cos x \\ -\sin x \end{pmatrix} = \begin{pmatrix} \cos^2 x & \frac{1}{2}\sin 2x - 1 \\ \frac{1}{2}\sin 2x + 1 & \sin^2 x \end{pmatrix}\begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}$$

同理可验证另一组解.

存在唯一性定理：方程组 (*) 满足 $\vec{y}(x_0) = \vec{y_0}$ 的解

$\vec{y} = \vec{y}(x)$ 在 $x\in(a.b)$ 上存在且唯一

其中 $x_0\in(a.b)$ 和 $y_0\in\mathbb{R}^n$ 是任意给定.

引理 6.1. 设 $\vec{y} = \vec{y_1}(x)$ 和 $\vec{y} = \vec{y_2}(x)$ 为 (*) 式 $\vec{f}\equiv 0$ 时的解

则线性组合 $\vec{y} = c_1\vec{y_1} + c_2\vec{y_2}$ 也是原方程解.

证明：代入验证即得

引理 6.2：设齐次方程 $\frac{d\vec{y}}{dx}=A\vec{y}$ 在 (a,b) 上所有解构成集合为 S，则 S 为线性空间，且 $\dim S=n$。

证明：由引理 6.1 知其为线性空间。

令 $x_0\in(a,b)$，则 $\forall\vec{y_0}\in\mathbb{R}^n$，在 S 中 $\exists 1$ $\vec{y}(x)$

s.t. $\vec{y}(x_0)=\vec{y_0}$　故可定义 $H:\vec{y_0}\to\vec{y}(x)$　$\mathbb{R}^n\to S$

显然 $\forall\vec{y}(x)\in S$ 有 $\vec{y}(x_0)\in\mathbb{R}^n$　$H(\vec{y}(x_0))=\vec{y}(x)$

故 H 为满射。

又 $\forall\vec{y_1^0},\vec{y_2^0}\in\mathbb{R}^n$，令 $\vec{y_1}(x)=H(\vec{y_1^0})$　$\vec{y_2}(x)=H(\vec{y_2^0})$

由唯一性知 $\vec{y_1}(x)\neq\vec{y_2}(x)$　$(a<x<b)$

当且仅当 $\vec{y_1^0}\neq\vec{y_2^0}$

从而 H 为一一映射　又易知 $H(c_1\vec{y_1^0}+c_2\vec{y_2^0})=c_1H(\vec{y_1^0})+c_2H(\vec{y_2^0})$

故有 H 为 $\mathbb{R}^n$ 到 S 的同构。

即 $S\cong\mathbb{R}^n$　从而 $\dim S=n$。

定理 6.1　设 $\frac{d\vec{y}}{dx}=A\vec{y}$ 在 (a,b) 上有 n 个线性无关解 $\vec{\varphi_1},\vec{\varphi_2}\cdots\vec{\varphi_n}$，则有原方程的通解为

$$\vec{y}=C_1\vec{\varphi_1}+\cdots+C_n\vec{\varphi_n}\quad\cdots\cdots(**)$$

注：若令 $\vec{y_k^0}\in\mathbb{R}^n$，$\vec{y_k}(x)=H(\vec{y_k^0})$，则 $\{\vec{y_k^0}\}$ $k=1,2\cdots m$ 在 $\mathbb{R}^n$ 中线性无关等价于 $\vec{y_1}(x)\cdots\vec{y_m}(x)$ 在 S 中线性无关。

即 $C_1\vec{y_1^0}+\cdots+C_m\vec{y_m^0}=0\Rightarrow C_1=\cdots=C_m=0$

等价于 $C_1\vec{y_1}(x)+\cdots+C_m\vec{y_m}(x)=0$

$\Rightarrow C_1=C_2=\cdots=C_m=0$

定理 6.1 的证明：由引理 6.2 知 $\{\vec{\varphi_1}\cdots\vec{\varphi_n}\}$ 为 S 的一个基，从而 $\vec{\varphi_1}\cdots\vec{\varphi_n}$ 张成的线性空间即为 S。

即 $(**)$ 式表示 $\frac{d\vec{y}}{dx}=A\vec{y}$ 的通解

称 $\frac{d\vec{y}}{dx} = A\vec{y}$ 的 n 个线性无关解为一个基本解组

假设已知 $\vec{y_1} \cdots \vec{y_n}$ 为 n 个解

$$\vec{y_1} = \begin{pmatrix} y_{11}(x) \\ \vdots \\ y_{n1}(x) \end{pmatrix} \quad \cdots \quad \vec{y_n} = \begin{pmatrix} y_{1n}(x) \\ \vdots \\ y_{nn}(x) \end{pmatrix}$$

定义：
$$W(x) = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}$$ 称为 Wronsky 行列式

引理 6.3 $\quad W(x) = W(x_0) e^{\int_{x_0}^{x} tr A(x) dx}$

证明：
$$\frac{dW}{dx} = \sum_{i=1}^{n} \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \\ y_{i1}' & \cdots & y_{in}' \\ \vdots & & \\ y_{n1} & \cdots & y_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} a_{ij} y_{j1} & \sum_{j=1}^{n} a_{ij} y_{j2} & \cdots & \sum_{j=1}^{n} a_{ij} y_{jn} \\ \vdots & \vdots & & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & \vdots & & \vdots \\ a_{ii} y_{i1} & a_{ii} y_{i2} & \cdots & a_{ii} y_{in} \\ \vdots & \vdots & & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} a_{ii} \cdot W = trA(x) \cdot W.$$

从而有 $\frac{dW}{dx} = trA(x) \cdot W$，解得

$$W(x) = W(x_0) e^{\int_{x_0}^{x} trA(x) dx}$$

推论6.1：$\vec{y}_1(x) \cdots \vec{y}_n(x)$ 线性相关等价于 $W(x) \equiv 0$.

证明：由引理6.3知 $W(x) \equiv 0$ 等价于

$\exists x_0$ s.t. $W(x_0) = 0$

令 $Y(x) = (y_{ij}(x))_{n\times n}$ 称为解矩阵

$$\frac{dY}{dx} = \left(\frac{dy_{ij}}{dx}\right)_{n\times n} = \left(\sum_{k=1}^{n} a_{ik}\, y_{kj}\right)_{n\times n}$$

$$= (a_{ij})_{n\times n}\,(y_{ij})_{n\times n} = A\cdot Y.$$

引理6.4. 若 $\Phi(x)$ 为 $\frac{d\vec{y}}{dx} = A\vec{y}$ 的一个基解矩阵

$\vec{\varphi}^*(x)$ 为 $\frac{d\vec{y}}{dx} = A\vec{y} + \vec{f}$ 的一个特解.

则原方程任一解 $\vec{\varphi}(x) = \Phi(x)\vec{c} + \vec{\varphi}^*(x)$

证明：显然 $\frac{d(\vec{\varphi}-\vec{\varphi}^*)}{dx} = A(\vec{\varphi}-\vec{\varphi}^*)$

即 $\vec{\varphi}(x) - \vec{\varphi}^*(x)$ 是齐次方程解.

从而 $\vec{\varphi}(x) - \vec{\varphi}^*(x) = \Phi(x)\vec{c}$.

引理6.5. 设 $\Phi(x)$ 为基解矩阵. 则若定义

$\varphi^*(x) = \Phi(x)\int_{x_0}^{x}\Phi^{-1}(s)f(s)\,ds$ 给出 $\frac{d\vec{y}}{dx} = A\vec{y} + \vec{f}$ 的特解.

证明：利用常数变易法. $\varphi^*(x) = \Phi(x)c(x)$ 有

$\Phi'(x)c(x) + \Phi(x)c'(x) = A(x)\Phi(x)c(x) + f(x)$

又由 $\Phi'(x) = A(x)\Phi(x)$, 故 $\Phi(x)c'(x) = f(x)$.

$\because \det\Phi(x) = W(x) \neq 0$. $\therefore c'(x) = \Phi^{-1}(x)f(x)$.

从而 $c(x) = \int_{x_0}^{x}\Phi^{-1}(s)f(s)\,ds$ 代回有

$\varphi^*(x) = \Phi(x)\int_{x_0}^{x}\Phi^{-1}(s)f(s)\,ds$

定理6.3 设$\Phi(x)$为基解矩阵，则原方程通解

$$\vec{y}=\Phi(x)\left(c+\int_{x_0}^{x}\Phi^{-1}(s)f(s)\,ds\right)$$ ，c为常向量

且满足初值条件$\vec{y}(x_0)=\vec{y}_0$的解为

$$\vec{y}=\Phi(x)\Phi^{-1}(x_0)y_0+\Phi(x)\int_{x_0}^{x}\Phi^{-1}(s)f(s)\,ds.$$

证明：由引理6.4, 6.5即得.

§6.2 常系数线性微分方程组

问题：如何找$\Phi(x)$？先考虑常系数情形.

$$\begin{cases}\dfrac{dx_1}{dt}=a_{11}x_1+a_{12}x_2\\[2mm]\dfrac{dx_2}{dt}=a_{21}x_1+a_{22}x_2\end{cases}\qquad A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$$ 为常数矩阵

$$\varphi_0=\begin{pmatrix}k_0\\k_1\end{pmatrix}=x_0\qquad \varphi_1(t)=x_0+\int_0^t Ax_0\,ds=x_0+tAx_0.$$

$$\varphi_2(t)=x_0+\int_0^t A\varphi_1(s)\,ds=x_0+tAx_0+\frac{t^2}{2}A^2x_0$$

一般的 $\varphi_n(t)=x_0\left(I+tA+\dfrac{t^2}{2}A^2+\cdots+\dfrac{t^n}{n!}A^n\right)$

问题：$\varphi_n(t)$的收敛性？

$$|\varphi_n-\varphi_m|\le\sum_{k=n}^{m}\left|\frac{t^k}{k!}A^kx_0\right|\le\sum_{k=n}^{m}\frac{t^k|A|^k}{k!}|x_0|$$

$$\le|x_0|\left(e^{t|A|}-\sum_{k=0}^{n}\frac{t^k|A|^k}{k!}\right)\rightrightarrows 0\quad(n\to\infty).$$

从而可设$\varphi_n(t)\rightrightarrows\varphi(t)$

$$\varphi_n(t)=x_0+tAx_0+\cdots+\frac{t^n}{n!}A^nx_0\triangleq T_nx_0.$$

则易知 $T_nx_0\rightrightarrows Tx_0$. 记 $T=e^{tA}$.

则 $e^{tA}\triangleq\displaystyle\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}$.

例1：计算 e^A，其中 A 定义为

(1) $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ (2) $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ (3) $\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$

解：(1) $e^{\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}} = \sum\limits_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix}$

(2) $e^{\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}} = e^{\lambda I} e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} = e^\lambda \cdot e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$

而 $e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} = \sum\limits_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^k = I + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

故原式 $= \begin{pmatrix} e^\lambda & 0 \\ e^\lambda & e^\lambda \end{pmatrix}$

(3) $e^{\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}} = e^{aI - b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = e^{aI} e^{-b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$

$e^{-b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \sum\limits_{k=0}^{\infty} \frac{1}{k!} (-1)^k b^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^k$

$= \sum\limits_{k=0}^{\infty} \frac{-1}{(2k+1)!} b^{2k+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum\limits_{k=0}^{\infty} \frac{1}{(2k)!} b^{2k} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= -\begin{pmatrix} 0 & \operatorname{sh} b \\ \operatorname{sh} b & 0 \end{pmatrix} + \begin{pmatrix} \operatorname{ch} b & 0 \\ 0 & \operatorname{ch} b \end{pmatrix} = \begin{pmatrix} \operatorname{ch} b & -\operatorname{sh} b \\ -\operatorname{sh} b & \operatorname{ch} b \end{pmatrix}$

例2：$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$，求 e^{xA}.

解：$e^{xA} = e^{xI} e^{xZ} = e^x \cdot e^{x\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

而 $e^{x\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = I + x \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

故有 $e^{xA} = e^x \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix}$

命题2：(1) 若 $AB = BA$，则有

$$e^{A+B} = e^A e^B.$$

(2) $\forall A \in M_{n\times n}$，$e^A$ 可逆，且 $(e^A)^{-1} = e^{-A}$.

(3) P 非奇异 则 $e^{PAP^{-1}} = P e^A P^{-1}$.

证明：(1) 考察方程 $\begin{cases} \dfrac{d\vec{X}}{dt} = (A+B)\vec{X} \\ \vec{X}\big|_{t=0} = \vec{X}_0 \end{cases}$

则存在唯一解 $\vec{X}(t) = e^{t(A+B)}\vec{X}_0$

另一方面，设 $\tilde{\vec{X}}(t) = e^{tA}e^{tB}\vec{X}_0$.

则 $\dfrac{d\tilde{\vec{X}}(t)}{dt} = (Ae^{tA}e^{tB} + Be^{tA}e^{tB})\vec{X}_0$

$= (A+B)e^{tA}e^{tB}\vec{X}_0 = (A+B)\tilde{\vec{X}}(t)$

从而有 $\vec{X}(t) = \tilde{\vec{X}}(t)$. 故 $e^{t(A+B)} = e^{tA}e^{tB}$.

令 $t=1$ 即有 $e^{A+B} = e^A e^B$.

(2) $e^A = \sum\limits_{m=0}^{\infty} \dfrac{A^m}{m!}$ $\quad e^{-A} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n A^n}{n!}$.

故 $e^A \cdot e^{-A} = \left(\sum\limits_{m=0}^{\infty} \dfrac{A^m}{m!}\right)\left(\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n A^n}{n!}\right)$

$= I + \sum\limits_{m=0}^{k} \dfrac{(-1)^{k-m}}{m!(k-m)!} A^k$

$= I + \dfrac{1}{k!}\sum\limits_{m=0}^{k} C_k^m (-1)^{k-m} A^k = I + O = I$.

(3) $e^{PAP^{-1}} = \sum\limits_{k=0}^{\infty} \dfrac{1}{k!}(PAP^{-1})^k$

$= \sum\limits_{k=0}^{\infty} \dfrac{1}{k!} PA^k P^{-1} = Pe^A P^{-1}$.

Jordan 标准形.

记 $J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$ $\quad J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix}$

由 $J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix} = \lambda_i I + \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$

$$e^{xJ_i} = e^{x\lambda_i I + x\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0\end{pmatrix}} = e^{x\lambda_i}\begin{pmatrix} 1 & x & \cdots & \frac{x^{n-1}}{(n-1)!} \\ & 1 & x & \vdots \\ & & \ddots & x \\ & & & 1 \end{pmatrix}$$

例 1：$\begin{cases} \dfrac{dx}{dt} = x - 5y \\ \dfrac{dy}{dt} = 2x - y \end{cases}$.

解：设 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} e^{\lambda t}$

$$\Rightarrow \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} r_1 \lambda e^{\lambda t} \\ r_2 \lambda e^{\lambda t} \end{pmatrix} = \begin{pmatrix} r_1 e^{\lambda t} - 5 r_2 e^{\lambda t} \\ 2r_1 e^{\lambda t} - r_2 e^{\lambda t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} r_1 \lambda = r_1 - 5r_2 \\ r_2 \lambda = 2r_1 - r_2 \end{cases} \Rightarrow \begin{cases} (1-\lambda) r_1 - 5r_2 = 0 \\ 2r_1 + (-1-\lambda) r_2 = 0 \end{cases}$$

$$\begin{vmatrix} 1-\lambda & -5 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i.$$

对 $\lambda_1 = 3i$，可取 $r_1 = 5$，$r_2 = 1-3i$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & e^{3it} \\ (1-3i) & e^{3it} \end{pmatrix} = \begin{pmatrix} 5\cos 3t + 5i\sin 3t \\ (1-3i)\cos 3t + (3+i)\sin 3t \end{pmatrix}$$

取共轭知

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5\cos 3t - 5i\sin 3t \\ (1+3i)\cos 3t + (3-i)\sin 3t \end{pmatrix}$$ 也是解.

由上述两解表达式知通解为

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 5\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 3t \\ \sin 3t - 3\cos 3t \end{pmatrix}$$

c_1, c_2 为常数.

例2：$\begin{cases} \dfrac{dy_1}{dx} = y_1 - y_2 \\ \dfrac{dy_2}{dx} = y_1 + 3y_2 \end{cases}$

解：$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-3 \end{vmatrix} = (\lambda-2)^2$.

故有 $\lambda_1 = \lambda_2 = 2$.　　设 $y_i = (a_i + b_i x)e^{2x}$ $(i=1,2)$

代入有 $\begin{cases} b_1 + b_2 = 0 \\ a_1 + a_2 + b_1 = 0 \end{cases}$.　从而若设 $\begin{cases} b_1 = c_1 \\ a_1 = c_2 \end{cases}$.

则有 $y_1 = (c_1 x + c_2)e^{2x}$, $y_2 = (-c_1 x - c_1 - c_2)e^{2x}$.

即 $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 x e^{2x} + c_2 e^{2x} \\ -c_1 x e^{2x} - (c_1 + c_2)e^{2x} \end{pmatrix}$

$$= c_1 \begin{pmatrix} xe^{2x} \\ -xe^{2x} - e^{2x} \end{pmatrix} + c_2 \begin{pmatrix} e^{2x} \\ -e^{2x} \end{pmatrix}$$

§6.3 高阶线性方程

本节讨论 n 阶线性微分方程式

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = f(x).$$

如果引进 $y_1 = y$, $y_2 = y'$ $\cdots$ $y_n = y^{(n-1)}$ ……(*)

则易见原式等价于下面的线性方程组.

$$\frac{d\vec{y}}{dx} = A(x)\vec{y} + \vec{f}(x)$$

其中 $\vec{y}=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ $\vec{f}(x)=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{pmatrix}$

且 $A(x)=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}$

从而当 $f(x)=0$ 时 $\dfrac{d\vec{y}}{dx}=A(x)\vec{y}$. ……(**)

假设函数组 $\varphi_1(x)$ … $\varphi_n(x)$ 为 n 个解

则由 (*) 式知 (**) 的解为

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_1'(x) \\ \vdots \\ \varphi_1^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} \varphi_2(x) \\ \varphi_2'(x) \\ \vdots \\ \varphi_2^{(n-1)}(x) \end{pmatrix} \cdots \begin{pmatrix} \varphi_n(x) \\ \varphi_n'(x) \\ \vdots \\ \varphi_n^{(n-1)}(x) \end{pmatrix}$$

Wronsky 行列式为

$$W(x)=\begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_n(x) \\ \varphi_1'(x) & \varphi_2'(x) & \cdots & \varphi_n'(x) \\ \vdots & \vdots & & \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & \cdots & \varphi_n^{(n-1)}(x) \end{vmatrix}$$

例 1: 若已知 $\varphi_1(x)$, $\varphi_2(x)$ 为 $y''+p(x)y'+q(x)y=0$ 的两个线性无关解. 求 $y''+p(x)y'+q(x)y=f(x)$ 的通解

解: 设 $y=C_1\varphi_1+C_2\varphi_2$ (C_1, C_2 为函数)

则 $y'=C_1\varphi_1'+C_2\varphi_2'+\varphi_1C_1'+\varphi_2C_2'$

在上式中令 $c_1'\varphi_1 + c_2'\varphi_2 = 0$ ---- ①

则有 $y' = c_1\varphi_1' + c_2\varphi_2'$

$y'' = c_1'\varphi_1' + c_2'\varphi_2' + c_1\varphi_1'' + c_2\varphi_2''$

从而 $y'' + py' + qy = f$

$\Rightarrow (c_1'\varphi_1' + c_2'\varphi_2') + p(c_1\varphi_1' + c_2\varphi_2') + q(c_1\varphi_1 + c_2\varphi_2)$

$+ c_1\varphi_1'' + c_2\varphi_2'' = f$

$\Rightarrow c_1'\varphi_1' + c_2'\varphi_2' + (\varphi_1'' + p\varphi_1' + q\varphi_1)c_1 + (\varphi_2'' + p\varphi_2' + q\varphi_2)c_2 = f$

从而有 $c_1'\varphi_1' + c_2'\varphi_2' = f(x)$ --- ②

由①. ② $\Rightarrow \begin{cases} c_1' = -\dfrac{\varphi_2 f}{W} \\ c_2' = \dfrac{\varphi_1 f}{W} \end{cases}$

解出 $c_1(x)$. $c_2(x)$ 代入 $y = c_1\varphi_1 + c_2\varphi_2$ 得

$$y = c_1\varphi_1 + c_2\varphi_2 + \int_{x_0}^{x} \frac{\varphi_1(s)\varphi_2(x) - \varphi_1(x)\varphi_2(s)}{\varphi_1(s)\varphi_2'(s) - \varphi_2(s)\varphi_1'(s)} f(s)\, ds$$

例2: $y'' - 4y = 0$.

解: $\lambda^2 - 4 = 0 \Rightarrow \lambda_1 = 2$. $\lambda_2 = -2$. 从而

$y = c_1 e^{2x} + c_2 e^{-2x}$ 即为通解.

例3: 求 $\inf \left\{ \int_0^1 (y')^2 dx \,\middle|\, \int_0^1 y^2 dx = 1.\ y(0) = y(1) = 0 \right\}$

解: $y_s(x) = y(x) + s\,v(x)$. $v \in C_0^\infty[0,1]$.

$$J_s(y) = \frac{\int_0^1 (y_s')^2 dx}{\int_0^1 y_s^2 dx} \quad \text{由} \left.\frac{d}{ds} J_s\right|_{s=0} = 0$$

$$\Rightarrow \frac{2\int_0^1 y(x)v(x)\,dx}{\int_0^1 y^2(x)\,dx} - \frac{\int_0^1 (y')^2 dx \cdot 2\int_0^1 yv\,dx}{\left(\int_0^1 y^2(x)\,dx\right)^2} = 0$$

记 $\lambda = \dfrac{\int_0^1 (y')^2 dx}{\int_0^1 y^2 dx}$. 则有 $\int_0^1 y'v' dx = \lambda \int_0^1 yv\, dx$

$$\Rightarrow (y'v)\Big|_0^1 - \int_0^1 y''v\, dx = \lambda \int_0^1 yv\, dx$$

$$\Rightarrow \int_0^1 (y'' + \lambda y) v\, dx = 0$$. 由 v 任意性知

$y'' + \lambda y = 0$. 从而 $y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

由 $y(0) = y(1) = 0$ 知 $c_1 = 0$. $\sin\sqrt{\lambda} = 0$.

从而有 $\sqrt{\lambda} = k\pi$. 即 $\lambda = k^2\pi^2$. $k \in \mathbb{Z}$.

于是有 $\lambda \geq \pi^2$. 易见 $y = \sin\pi x$ 时

$y' = \pi\cos\pi x$. 且 $\int_0^1 (\pi\cos\pi x)^2 dx = \frac{\pi^2}{2}$.

$\int_0^1 (\sin\pi x)^2 dx = \frac{1}{2}$.

此时 $\lambda = \dfrac{\int_0^1 (\pi\cos\pi x)^2 dx}{\int_0^1 (\sin\pi x)^2 dx} = \dfrac{\frac{\pi^2}{2}}{\frac{1}{2}} = \pi^2$

即等号可以取到. 从而

$$\inf \left\{ \int_0^1 (y')^2 dx \,\middle|\, \int_0^1 y^2 dx = 1 ,\ y(0) = y(1) = 1 \right\} = \pi^2$$

第七章 矩形与圆盘的 Dirichlet 问题

例1：$\begin{cases} \Delta u + \lambda u = 0 & \text{in } [0,2]\times[0,1] \\ u = 0 & \text{on } \partial\square \end{cases}$

解：设 $u = X(x)Y(y)$. 则 $\Delta u = X''Y + XY''$.

$\Delta u + \lambda u = 0 \Rightarrow X''Y + XY'' + \lambda XY = 0$

$\Rightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} + \lambda = 0$.

设 $\dfrac{X''}{X} = -\alpha$. $\dfrac{Y''}{Y} = -\beta$. 则 $\alpha + \beta = \lambda$.

从而 $X = c_1 \cos(\sqrt{\alpha}\, x) + c_2 \sin(\sqrt{\alpha}\, x)$

$Y = c_3 \cos(\sqrt{\beta}\, y) + c_4 \sin(\sqrt{\beta}\, y)$.

易见 $X(0) = X(2) = 0$. $Y(0) = Y(1) = 0$

代入有 $\begin{cases} X = c_2 \sin(\sqrt{\alpha}\, x) \\ Y = c_4 \sin(\sqrt{\beta}\, y) \end{cases} \Rightarrow \begin{cases} c_2 \sin(2\sqrt{\alpha}) = 0 \\ c_4 \sin(\sqrt{\beta}) = 0 \end{cases}$

$\Rightarrow \begin{cases} 2\sqrt{\alpha} = l\pi \\ \sqrt{\beta} = k\pi \end{cases} \Rightarrow \begin{cases} \alpha = \dfrac{l^2\pi^2}{4} \\ \beta = k^2\pi^2 \end{cases} \quad (k, l \in \mathbb{Z}^+)$.

从而只有 $\lambda = \left(\dfrac{l^2}{4} + k^2\right)\pi^2$ 时方程有非零解

$u(x,y) = C_{l,k} \sin\left(\dfrac{l x \pi}{2}\right) \sin(k y \pi) \quad k, l \in \mathbb{Z}^+$

例2：$\begin{cases} \Delta u + \lambda u = 0 & \text{in } B_1(0) \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial B_1(0) \end{cases}$

解：$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \Delta u = u_{rr} + \dfrac{1}{r} u_r + \dfrac{1}{r^2} u_{\theta\theta}$.

设 $u(r,\theta) = R(r)\Theta(\theta)$. 则有

$R''\Theta + \dfrac{1}{r} R'\Theta + \dfrac{1}{r^2} R\Theta'' = -\lambda R\Theta$

$$\Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

$$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta}$$

设 $\frac{\Theta''}{\Theta} = -\mu$. 则 $\Theta = c_1 \cos(\sqrt{\mu}\theta) + c_2 \sin(\sqrt{\mu}\theta)$

由 $\Theta(0) = \Theta(2\pi)$ 知 $\sqrt{\mu} \in N$. 即 $\mu = n^2 \quad (n \in N)$

从而代入得 $r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda r^2 = n^2$

$$\Rightarrow r^2R'' + rR' + (\lambda r^2 - n^2)R = 0$$

此即 Bessel 方程，可通过幂级数求解.

类似的，对于球上的 Dirichlet 问题可利用球坐标求解

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

其中要涉及到 Lengendre 函数.

第八章 稳定性

§8.1 例题与定理

例1：
$$\begin{cases} \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0 \\ y(0) = 0,\quad y'(0) = 1 \end{cases}$$

解： $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda_1 = -1+i,\ \lambda_2 = -1-i$.

$y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$，代入得

$y(t) = e^{-t}\sin t \qquad y'(t) = e^{-t}(\cos t - \sin t)$.

下面作初值扰动 $\tilde{y}(0) = \eta_1$，$\tilde{y}'(0) = 1+\eta_1$.

则类似可解出 $\tilde{y}(t) = e^{-t}[\eta_1\cos t + (1+\eta_1+\eta_2)\sin t]$

则 $\tilde{y}'(t) = e^{-t}[(1+\eta_2)\cos t - (1+\eta_2+2\eta_1)\sin t]$

易知 $|\tilde{y}(t) - y(t)| = |\eta_1 e^{-t}\cos t + (\eta_1+\eta_2)e^{-t}\sin t|$

$\leq (2|\eta_1| + |\eta_2|)e^{-t} \to 0 \quad (t \to +\infty)$.

例2：
$$\begin{cases} \frac{dy_1}{dt} = -2y_1 - 4y_2 + 1 + 4t \\ \frac{dy_2}{dt} = -y_1 + y_2 + \frac{3}{2}t^2 \end{cases} \quad \text{且} \quad \begin{cases} y_1(0) = 1 \\ y_2(0) = -1 \end{cases}$$

解： 原方程解为
$$\begin{cases} y_1 = e^{2t} + t^2 + t \\ y_2 = -e^{2t} - \frac{t^2}{2} \end{cases}$$

下面作初值扰动. $\tilde{y}(0) = \begin{pmatrix} \tilde{y}_1(0) \\ \tilde{y}_2(0) \end{pmatrix} = \begin{pmatrix} 1+\eta_1 \\ -1+\eta_2 \end{pmatrix}$

可解出
$$\begin{cases} \tilde{y}_1 = \frac{\eta_1 - 4\eta_2 + 5}{5}e^{2t} + \frac{4(\eta_1-\eta_2)}{5}e^{-3t} + t + t^2 \\ \tilde{y}_2 = -\frac{\eta_1 - 4\eta_2 + 5}{5}e^{2t} + \frac{\eta_1+\eta_2}{5}e^{-3t} - \frac{t^2}{2} \end{cases}$$

$$|\tilde{y}_1(t)-y_1(t)| = \left|\frac{\eta_1-4\eta_2}{5}e^{2t}+\frac{4(\eta_1-\eta_2)}{5}e^{-3t}\right|$$

$$|\tilde{y}_2(t)-y_2(t)| = \left|-\frac{\eta_1-4\eta_2}{5}e^{2t}+\frac{\eta_1+\eta_2}{5}e^{-3t}\right|$$

$t\to+\infty$ 时 $|\tilde{y}_i(t)-y_i(t)|\to+\infty$，从而不稳定．

定义：方程 $\frac{d\vec{x}}{dt}=\vec{f}(t,x)$ 的解 $\vec{x}=\vec{\varphi}(t)$ 满足

$\forall\varepsilon>0$，$\exists\delta=\delta(\varepsilon)>0$，只要 $|\vec{x_0}-\vec{\varphi}(t_0)|<\delta$

原方程以 $\vec{x}(t_0)=\vec{x_0}$ 为初值的解 $\vec{x}(t,t_0,\vec{x_0})$ 满足

$|\vec{x}(t,t_0,\vec{x_0})-\vec{\varphi}(t)|<\varepsilon \quad (t\geq t_0)$

则称 $\vec{\varphi}(t)$ 是稳定的．假如 $\vec{\varphi}(t)$ 稳定，且 $\exists\delta_1\leq\delta$

s.t. 只要 $|\vec{x_0}-\vec{\varphi}(t_0)|<\delta_1$，则有 $\lim\limits_{t\to+\infty}|\vec{x}(t,t_0,\vec{x_0})-\vec{\varphi}(t)|=0$．

则称 $\vec{\varphi}(t)$ 是渐近稳定的．

定理 8.1：设 $\frac{d\vec{x}}{dt}=A(t)\vec{x}$ 中的 $A(t)$ 为常矩阵

则 (1) 零解是渐近稳定的 $\Longleftrightarrow$ A 的所有特征值 $\lambda_1\cdots\lambda_n$

满足 $\mathrm{Re}\,\lambda_i<0$． $(1\leq i\leq n)$

(2) 零解是稳定的 $\Longleftrightarrow$ $\mathrm{Re}\,\lambda_i\leq 0$． $(1\leq i\leq n)$

(3) 零解是不稳定的 $\Longleftrightarrow$ $\exists i$．s.t．$\mathrm{Re}\,\lambda_i>0$

或者 $\exists j$ s.t. $\mathrm{Re}\,\lambda_j=0$ 且 λ_j 对应的 Jordan 块高于一阶

证明从略

§8.2 Liyapanov 第二方法

考察 $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$ $\vec{x} \in \mathbb{R}^n$. $\vec{f}(\vec{x}) = (f_1(x) \cdots f_n(x))$

满足初值问题解的存在唯一性.

设 $V(x)$ 在 $|x| \le M$ 上有定义. 且连续可微.

条件: I. $V(0) = 0$. $V(x) > 0$ $(x \ne 0)$.

II. $\left.\frac{dV}{dt}\right|_{\frac{dx}{dt}=f(x)} = \frac{\partial V}{\partial x_1} f_1 + \cdots + \frac{\partial V}{\partial x_n} f_n < 0$ $(x \ne 0)$

II*. $\left.\frac{dV}{dt}\right|_{\frac{dx}{dt}=f(x)} \le 0$

III. $\left.\frac{dV}{dt}\right|_{\frac{dx}{dt}=f(x)} > 0$ $(x \ne 0)$.

若: I 和 II 成立. 则零解渐近稳定

I 和 II* 成立. 则零解稳定

I 和 III 成立. 则零解不稳定

例1:
$$\begin{cases} \frac{dx}{dt} = -x + y \\ \frac{dy}{dt} = x\cos t - y \end{cases}$$

解: 令 $V = \frac{1}{2}(x^2 + y^2)$ 满足条件 I.

$$\frac{dV}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = x(y-x) + y(x\cos t - y)$$

$$= -x^2 - y^2 + xy(1+\cos t) \le 0$$ 满足 II*

当 $t = 2k\pi$ 且 $x = y$ 时可取等号. $(k \in \mathbb{Z})$

从而零解是稳定的.

例2: $\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\sin\varphi$

解：令 $x=\varphi \quad \frac{dx}{dt}=y$.

则 $\begin{cases} \frac{dx}{dt}=y \\ \frac{dy}{dt}=-\frac{g}{L}\sin x \end{cases}$.

由 $y\frac{dy}{dt}=-\frac{g}{L}y\sin x=-\frac{g}{L}\frac{dx}{dt}\sin x$

$\Rightarrow y\,dy=-\frac{g}{L}\sin x\,dx$

$\Rightarrow d\left(\frac{1}{2}y^2+\frac{g}{L}(1-\cos x)\right)=0$.

令 $V(x,y)=\frac{g}{L}(1-\cos x)+\frac{1}{2}y^2$.

则有 V 满足 I, II*. 从而零解是稳定的.

偏微分方程稳定性. (1985年)

$\begin{cases} u_t=\Delta u+u^p & \mathbb{R}^+\times\mathbb{R}^n \\ u|_{t=0}=u_0(x) \end{cases}$ 的正解的稳定性.

其中 $p\in\left(1,\frac{n+2}{n-2}\right)$.

第九章 Sturm比较定理

讨论 $y'' + p(x)y' + q(x) = 0$ ……(*)

其中 $p(x), q(x) \in C(J)$，J为区间.

引理 9.1. (*)的非零解在J内的零点孤立.

证明：$y=\varphi(x)$ $(x\in J)$ 为(*)的解（φ不恒为0）

设$\exists x_0 \in J$，在J中有$\{x_n\}$满足 $x_n \to x_0$.

$\varphi(x_n)=0$ 且 $\varphi(x_0)=0$.

故 $$\varphi'(x_0) = \lim_{n\to\infty} \frac{\varphi(x_n)-\varphi(x_0)}{x_n - x_0} = 0.$$

从而解$\varphi(x)$满足初值 $\varphi(x_0)=0$，$\varphi'(x_0)=0$

由解的唯一性知 $\varphi(x)\equiv 0$，矛盾.

定理 9.1. 设φ_1, φ_2为(*)式的两个解（不恒为零），则有

(1) φ_1, φ_2线性相关 $\Longleftrightarrow$ φ_1, φ_2有相同零点.

(2) φ_1, φ_2线性无关 $\Longleftrightarrow$ 它们的零点是互相交错的.

证明：(1) 若φ_1, φ_2线性相关，则$\exists c\neq 0$ s.t. $\varphi_1 = c\varphi_2$.

则显然 $\varphi_1 = 0 \Longleftrightarrow \varphi_2 = 0$.

反之，若$\exists x_0\in J$ s.t. $\varphi_1(x_0)=\varphi_2(x_0)$，则

$W(x_0)=0$，故$W(x)\equiv 0$，即线性相关.

(2) 若φ_1, φ_2线性无关，则无相同零点.

设x_1, x_2为φ_1的两个相邻零点，不妨设 $\varphi_1(x)>0$ in (x_1, x_2)

从而 $\varphi_1'(x_1)\geq 0$，$\varphi_1'(x_2)\leq 0$. $\because$ $\varphi_1(x)$为非零解

从而有 $\varphi_1'(x_1)>0$，$\varphi_1'(x_2)<0$. ……(**)

又 $\varphi_2(x_1)\varphi_2(x_2)\neq 0$. 不妨设 $\varphi_1(x_1)\varphi_2(x_2)>0$.

另一方面 $\because$ $W(x)\neq 0$ in J

故 $W(x_1)W(x_2)>0$. 而

$$W(x_1) = \begin{vmatrix} \varphi_1(x_1) & \varphi_2(x_1) \\ \varphi_1'(x_1) & \varphi_2'(x_1) \end{vmatrix} = -\varphi_2(x_1)\varphi_1'(x_1)$$

同理 $W(x_2) = -\varphi_2(x_2)\varphi_1'(x_2)$

故有 $\varphi_2(x_1)\varphi_2(x_2)\varphi_1'(x_1)\varphi_1'(x_2) > 0 \Rightarrow \varphi_1'(x_1)\varphi_1'(x_2) > 0$

与 (**) 矛盾. 从而 $\varphi_2(x_1)\varphi_2(x_2) < 0$.

故 $\varphi_2(x)$ 在 (x_1, x_2) 有根 $\tilde{x}_1$. 从而若 $\exists\, \tilde{x}_2 \neq \tilde{x}_1$

s.t. $\varphi_2(\tilde{x}_1) = \varphi_2(\tilde{x}_2) = 0$. 且 $x_1 < \tilde{x}_1 < \tilde{x}_2 < x_2$

则同上有 $\exists\, \tilde{x} \in (\tilde{x}_1, \tilde{x}_2)$, s.t. $\varphi_1(\tilde{x}) = 0$. 矛盾

故 φ_1、φ_2 的零点是交错的.

反之. 若 φ_1, φ_2 零点是相互交错的. 则无公共零点.

故由 (1) 知 φ_1, φ_2 线性无关.

分析零点比较技巧: $v'(x) + p(x)v(x) \geq 0$, $x \in (x_1, x_2)$

若 $p(x) \equiv 0$. 则有 $v'(x) \geq 0 \Rightarrow v(x_1) \geq v(x_2)$.

若 $p(x) \neq 0$. 则 $w(x) \triangleq v(x)\, e^{\int_{x_1}^{x} p(t)\,dt}$

$w'(x) = v'(x)\, e^{\int_{x_1}^{x} p(t)\,dt} + p(x)\,v(x)\, e^{\int_{x_1}^{x} p(t)\,dt} \geq 0$.

故 $w(x_2) = v(x_2)\, e^{\int_{x_1}^{x_2} p(t)\,dt} \geq w(x_1) = v(x_1)$

定理 9.2. $y'' + p(x)y' + Q(x)y = 0$ ⋯ ①

$y'' + p(x)y' + R(x)y = 0$ ⋯ ②

其中 $P, Q, R \in C(J)$. $R(x) \geq Q(x)$ $(x \in J)$

$\varphi(x)$ 为 ① 的解. x_1, x_2 为相邻零点. 则 ② 的解 $\psi(x)$

在 x_1、x_2 之间至少有一个零点 x_0.

证明: 由 $\varphi(x_1) = \varphi(x_2) = 0$. 不妨设 $\varphi(x) > 0$ in (x_1, x_2)

则 $\varphi'(x_1) > 0$. $\varphi'(x_2) < 0$. 反设 $\psi(x) > 0$ in $[x_1, x_2]$.

则由 $\begin{cases} \varphi'' + p\varphi' + Q\varphi = 0 \\ \psi'' + p\psi' + R\psi = 0 \end{cases}$

$\Rightarrow \begin{cases} \psi\varphi'' + p\varphi'\psi + Q\varphi\psi = 0 & \cdots ③ \\ \varphi\psi'' + p\varphi\psi' + R\varphi\psi = 0 & \cdots ④ \end{cases}$

定义 $V(x) = \psi(x)\varphi'(x) - \varphi(x)\psi'(x)$

③ − ④ $\Rightarrow \psi\varphi'' - \varphi\psi'' + PV + (Q-R)\varphi\psi = 0$

故有 $V' + PV = (R-Q)\varphi\psi \geq 0$

由之前分析有 $V(x_2)\, e^{\int_{x_1}^{x_2} p(x)\,dx} \geq V(x_1)$ …⑤

又 $\begin{cases} V(x_1) = \psi(x_1)\varphi'(x_1) \\ V(x_2) = \psi(x_2)\varphi'(x_2) \end{cases} \Rightarrow \begin{cases} V(x_1) > 0 \\ V(x_2) < 0 \end{cases}$ 与⑤矛盾

从而 $\psi(x) > 0$ in $[x_1, x_2]$ 不能成立.

同理有 $\psi(x) < 0$ in $[x_1, x_2]$ 也不能成立.

故 ψ 在 $[x_1, x_2]$ 中至少有一个零点.

即 $\exists\, x_0 \in [x_1, x_2]$ s.t. $\psi(x_0) = 0$.

注：定理 9.2 即 Sturm − Liouville 比较定理

第十章 首次积分

§10.1 定义与例子

先看两个常微分方程首次积分的例子

例1： $x''+1=0$

则 $x'x''+x'=0 \Rightarrow \left[\frac{1}{2}(x')^2\right]'+x'=0$

$\Rightarrow \frac{1}{2}(x')^2+x\equiv \text{const}$

若记 $P=\frac{1}{2}(x')^2+x$，则 P 为守恒量

例2： $$\frac{d^2x}{dt^2}=-\frac{g}{l}\sin x$$

易见 $$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2-\frac{g}{l}\cos x\right)$$

$$=\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}+\frac{g}{l}\sin x\,\frac{dx}{dt}=0$$

若记 $P=\frac{1}{2}\left(\frac{dx}{dt}\right)^2-\frac{g}{l}\cos x$，则 P 为守恒量

定义：函数 $V(x,y_1\cdots y_n)\in C^1(G)$

若沿方程 $\frac{dy_i}{dx}=f_i(x,y_1\cdots y_n)$ 的任一积分曲线

Γ： $y=y_1(x)\ \cdots\ y=y_n(x)\quad (x\in J)$

满足 $V(x,y_1(x)\cdots y_n(x))\equiv\text{const}\quad (x\in J)$

则称 $V(x,y_1\cdots y_n)\equiv c$ 为在 G 内的首次积分

例3：(1) $\Delta u=-1$ in $\Omega\subset\mathbb{R}^2$

定义 $P=\frac{1}{2}|\nabla u|^2+au$　确定 a 的值

使得 $\Delta P\geq 0$　（从而 $\max\limits_{\Omega}P=\max\limits_{\partial\Omega}P$）

(2) 若 $u=0$ on ∂B_1，对 (1) 中的 a

求出 P 的值

解：(1) $P_i = u_l u_{li} + a u_i$

$P_{ii} = u_{li}^2 + u_l u_{lii} + a u_{ii}$

$\Rightarrow \Delta P = |D^2u|^2 + a\Delta u = |D^2u|^2 - a$

$\geq u_{11}^2 + u_{22}^2 - a \geq \frac{1}{2}(\Delta u)^2 - a = \frac{1}{2} - a$.

取 $a=\frac{1}{2}$ 即可有 $\Delta P \geq 0$.

(2) 易见 $u(x_1,x_2) = \frac{1}{4} - \frac{1}{4}(x_1^2+x_2^2)$ 是解

$u_i = -\frac{x_i}{2}$ $\quad u_{ii} = -\frac{1}{2}$

$P = \frac{1}{2}|\nabla u|^2 + \frac{u}{2} = \frac{1}{2}\cdot\frac{x_1^2+x_2^2}{4} + \frac{1}{8}\left[1-(x_1^2+x_2^2)\right] = \frac{1}{8}$

例4：求 $u''+f(u)=0$ 的首次积分

解：$u'u''+f(u)u'=0$. 记 $F(u)=\int_0^u f(s)\,ds$.

则 $\left(\frac{1}{2}(u')^2+F(u)\right)'=0$. 令 $P=\frac{1}{2}(u')^2+F(u)$.

则有 $P\equiv \text{const}$. 即 P 为首次积分

例5：$\Delta u+\lambda u=0$ in $\Omega\subset\mathbb{R}^2$

令 $P=\frac{1}{2}|\nabla u|^2+\frac{\lambda}{4}u^2$. 求证 $\Delta P\geq -\frac{\lambda}{2}|\nabla u|^2$

证明：$P_i = u_l u_{li} + \frac{\lambda}{2} u u_i$

$P_{ii} = u_{li}^2 + u_l u_{lii} + \frac{\lambda}{2}u_i^2 + \frac{\lambda}{2} u u_{ii}$

$\Delta P = |D^2u|^2 + Du\cdot(-\lambda Du) + \frac{\lambda}{2}|Du|^2 - \frac{\lambda^2}{2}u^2$

$\geq \frac{1}{2}(\Delta u)^2 - \frac{\lambda}{2}|Du|^2 - \frac{\lambda^2}{2}u^2$

$= -\frac{\lambda}{2}|\nabla u|^2$ $\qquad$ 证毕.

§10.2 椭圆方程 极值原理

本节为下节的平均曲率方程作一些准备工作

定理1：设 $|b_i(x)| \le M$，u 满足

$\Delta u + b_i(x) u_i \ge 0$ in $\Omega \subset \mathbb{R}^n$　Ω 为有界区域

则有 $\max\limits_{\Omega} u = \max\limits_{\partial\Omega} u$.

证明：令 $v = u + \varepsilon e^{\alpha x_1}$（$\varepsilon, \alpha$ 待定）

则 $v_i = u_i + \varepsilon\alpha e^{\alpha x_1}\delta_{1i}$

$v_{ii} = u_{ii} + \varepsilon\alpha^2 e^{\alpha x_1}\delta_{1i}\delta_{1i}$

$\Delta v = \Delta u + \varepsilon\alpha^2 e^{\alpha x_1}$　故有

$\Delta v + b_i v_i = \Delta u + b_i u_i + \varepsilon\alpha^2 e^{\alpha x_1} + \varepsilon\alpha b_1 e^{\alpha x_1}$

$\ge \alpha^2 \varepsilon e^{\alpha x_1} + b_1 \alpha\varepsilon e^{\alpha x_1} > 0$　（取 $\alpha > M$）

从而若 v 在 $x_0 \in \Omega^0$ 取极大值.

则 $\Delta v + b_i v_i\big|_{x_0} = \Delta v(x_0) \le 0$　矛盾

故有 $\max\limits_{\Omega} v \le \max\limits_{\partial\Omega} v$

故 $\max\limits_{\Omega} u \le \max\limits_{\Omega} v \le \max\limits_{\partial\Omega} u + \varepsilon \max\limits_{\partial\Omega} e^{\alpha x_1}$

$\le \max\limits_{\partial\Omega} u + \varepsilon B$　令 $\varepsilon \to 0$ 有

$\max\limits_{\Omega} u \le \max\limits_{\partial\Omega} u \Rightarrow \max\limits_{\Omega} u = \max\limits_{\partial\Omega} u$

定理2：设 $a_{ij}(x)$ 满足 $\exists\, 0 < \lambda < \Lambda$. $\forall \xi \in \mathbb{R}^{1\times n}$

有 $\lambda|\xi|^2 \le \sum\limits_{i,j} a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$

且 $|b_i(x)| \le M$ in Ω　Ω 为有界区域.

u 满足 $\sum\limits_{i,j} a_{ij} u + \sum\limits_i b_i u_i \ge 0$

则有 $\max\limits_{\Omega} u = \max\limits_{\partial\Omega} u$

证明：令 $v = u + \varepsilon e^{\alpha x_1}$，则 $v_i = u_i + \varepsilon\alpha e^{\alpha x_1}\delta_{1i}$

$v_{ij} = u_{ij} + \varepsilon\alpha^2 e^{\alpha x_1}\delta_{1i}\delta_{1j}$ 从而有

$$\sum_{i,j} a_{ij}v_{ij} = \sum_{i,j} a_{ij}u_{ij} + \varepsilon\alpha^2 e^{\alpha x_1}\cdot a_{11}$$

$$\sum_i b_i v_i = \sum_i b_i u_i + \varepsilon\alpha e^{\alpha x_1}\cdot b_1$$

$$\Rightarrow \sum_{i,j} a_{ij}v_{ij} + \sum_i b_i v_i = \sum_{i,j} a_{ij}u_{ij} + \sum_i b_i u_i$$

$$+\varepsilon\alpha e^{\alpha x_1}(\alpha a_{11} + b_1) \geq \varepsilon\alpha e^{\alpha x_1}(\alpha a_{11} + b_1)$$

$$\geq \varepsilon\alpha e^{\alpha x_1}(\alpha\lambda - M) > 0 \quad \left(\text{取 } \alpha > \frac{M}{\lambda}\right)$$

从而若 v 在 $x_0 \in \Omega^0$ 取极大值

则有 $v_i(x_0) = 0$，且 $D^2v(x_0)$ 半负定

从而 $\left.\sum_{i,j} a_{ij}v_{ij} + \sum_i b_i v_i\right)_{x_0} \leq 0$ 矛盾

故有 $\max\limits_{\Omega} u \leq \max\limits_{\Omega} v \leq \max\limits_{\partial\Omega} v$

$\leq \max\limits_{\partial\Omega} u + \varepsilon \max\limits_{\partial\Omega} e^{\alpha x_1} \leq \max\limits_{\partial\Omega} u + \varepsilon B$.

令 $\varepsilon \to 0$ 有 $\max\limits_{\Omega} u \leq \max\limits_{\partial\Omega} u \Rightarrow \max\limits_{\Omega} u = \max\limits_{\partial\Omega} u$.

§10.3 平均曲率方程的首次积分

定义：方程 $D_i\left(\dfrac{u_i}{\sqrt{1+|Du|^2}}\right) = f(x)$ 称为平均曲率方程

例1：$\dfrac{u''}{\sqrt{1+(u')^2}^{\,3}} = 1$ （一维情形）

解：原式 $\Rightarrow u'u'' = u'\left(1+(u')^2\right)^{\frac{3}{2}}$

$$\Rightarrow \frac{u'u''}{\left(1+(u')^2\right)^{\frac{3}{2}}} - u' = 0$$

$$\Rightarrow d\left(\frac{1}{\sqrt{1+(u')^2}} - u\right) = 0 \Rightarrow \frac{1}{\sqrt{1+(u')^2}} - u = \text{const}$$

若令 $P(u,u') = \dfrac{1}{\sqrt{1+(u')^2}} - u$，则 $P \equiv const$

例2：$\sum_{i=1}^{2} D_i \left(\dfrac{u_i}{\sqrt{1+|\nabla u|^2}} \right) = -2$ （二维情形）

令 $P = 2 - \dfrac{2}{\sqrt{1+|\nabla u|^2}} + 2\alpha u$ 求 α 的值使得 P 满足极值原理

解：定义 $a_{ij} = (1+|\nabla u|^2)\delta_{ij} - u_i u_j$

则有 $a_{ij}u_{ij} = (1+|\nabla u|^2)\Delta u - u_i u_j u_{ij}$

令 $w = \sqrt{1+|\nabla u|^2}$ 则由原方程有

$$\frac{\Delta u}{w} - \frac{u_i u_j u_{ij}}{w^3} = -2$$

从而 $a_{ij}u_{ij} = w^2 \Delta u - u_i u_j u_{ij} = -2w^3$

又易见 $w_i = \dfrac{1}{w} u_k u_{ki}$ 且 $P = 2 - \dfrac{2}{w} + 2\alpha u$

则 $P_i = \dfrac{2w_i}{w^2} + 2\alpha u_i = \dfrac{2}{w^3} u_k u_{ki} + 2\alpha u_i$

$$P_{ij} = \frac{2 w_{kj} w_{ki} + 2 u_k u_{kij}}{w^3} - \frac{6 u_k u_{ki} u_l u_{lj}}{w^5} + 2\alpha u_{ij}$$

设 $P_i(x_0) = 0$，在 x_0 处旋转坐标系，使得有

$u_1 = |\nabla u|$　$u_2 = 0$，则有（在 x_0 处）

$a_{11} = (1+u_1^2) - u_1^2 = 1$　$a_{12} = 0$，$a_{21} = 0$

且 $a_{22} = (1+u_1^2) - u_2^2 = 1 + u_1^2 = w^2$

由 $P_1(x_0) = 0 \Rightarrow \dfrac{2}{w^3} u_1 u_{11} + 2\alpha u_1 = 0 \Rightarrow u_{11} = -\alpha w^3$

$P_2(x_0) = 0 \Rightarrow \dfrac{2}{w^3} u_1 u_{12} = 0 \Rightarrow u_{12} = 0$

又由 $a_{ij}u_{ij} = u_{11} + w^2 u_{22} = -2w^3 \Rightarrow u_{22} = (\alpha - 2)w$

而 $(a_{ij}u_{ij})_1 = -2(w^3)_1 \Rightarrow a_{ij}u_{ij1} = -a_{ij,1}u_{ij} - 2(w^3)_1$

$\Rightarrow a_{11}u_{111} + a_{22}u_{221} = -(a_{11})_1 u_{11} - (a_{22})_1 u_{22} - 2(a_{12})_1 u_{12} - 6w^2 w_1$

$$= -(1+u_2^2)_1 u_{11} - (1+u_1^2)_1 u_{22} - 2(a_{12})_1 u_{12} - 6 w u_k u_{k1}$$

$$= -2u_2 u_{12} u_{11} - 2u_1 u_{11} u_{22} - 2(a_{12})_1 u_{12} - 6w u_1 u_{11} - 6w u_2 u_{12}$$

$$= -2u_1 u_{11} u_{22} - 6w u_1 u_{11} = \alpha w^3 u_1 \left(2(\alpha-2)w + 6w\right)$$

$$= (2\alpha^2 + 2\alpha) w^4 u_1 \quad \cdots\cdots (*)$$

即 $u_{111} + w^2 u_{122} = (2\alpha^2 + 2\alpha) w^4 u_1$

则 $$a_{ij}P_{ij} = \frac{2a_{ij}u_{ki}u_{kj}}{w^3} + \frac{2u_1 a_{ij} u_{1ij}}{w^3} - \frac{6u_1^2 u_{1i} u_{1j} a_{ij}}{w^5} + 2\alpha a_{ij}u_{ij} \triangleq A + B + C + D.$$

$$A = 2w^{-3}(u_{11}^2 + w^2 u_{22}^2) = 2w^{-3}(\alpha^2 w^6 + (\alpha-2)^2 w^4)$$

$$= 2\alpha^2 w^3 + 2(\alpha-2)^2 w$$

$$B = 2w^{-3} u_1 (u_{111} + w^2 u_{122}) \overset{(*)}{=} (4\alpha^2 + 4\alpha) w u_1^2$$

$$C = -6w^{-5} u_1^2 (u_{11}^2 + w^2 u_{12}^2) = -6\alpha^2 w u_1^2$$

$$D = 2\alpha(-2w^3) = -4\alpha w^3.$$

从而 $A + B + C + D = 2w\left[\alpha^2 w^2 + (\alpha-2)^2 - 2\alpha w^2 + w u_1^2(4\alpha - 2\alpha^2)\right]$ 再由 $w^2 = 1 + |\nabla u|^2 = 1 + u_1^2$.

$$= 2w\left[\alpha^2(1+u_1^2) + (\alpha-2)^2 - 2\alpha(1+u_1^2) + u_1^2(2\alpha - \alpha^2)\right]$$

$$= 2w\left(\alpha^2 + (\alpha-2)^2 - 2\alpha\right) = 2w(\alpha-2)(2\alpha-2)$$

令上式 $= 0$ 有 $\alpha = 1$ 或 $\alpha = 2$. 此时 $a_{ij}P_{ij} = 0$.

从而满足极值原理.

故 $$P = 2 - \frac{2}{\sqrt{1+|\nabla u|^2}} + 2u$$

或 $$P = 2 - \frac{2}{\sqrt{1+|\nabla u|^2}} + 4u$$

第十一章　一阶偏微分方程

§11.1　特征线法

定义：(x,t) 空间中的一条曲线称为特征线

如果解沿此曲线为常数或满足常微分方程.

例1：$\begin{cases} u_t + a u_x = 0 & (x,t)\in \mathbb{R}\times\mathbb{R}^+ \\ u|_{t=0} = \varphi(x) & \varphi\in C^1(\mathbb{R}) \end{cases}$

解：$\dfrac{dx}{dt} = a \Rightarrow x = at + x_0$.（特征线）

令 $u = u(x(t), t)$. 则有

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = a u_x + u_t = 0 .$$

从而 $u(x(t), t) = u(x(0), 0) = \varphi(x(0))$

$= \varphi(x-at) \Rightarrow u = \varphi(x-at)$

例2：$\begin{cases} u_t + a u_x = bu + f & \text{其中 } a \text{ 为常数} \\ u|_{t=0} = \varphi(x) & b = b(t,x) \end{cases}$

解：$x(t) = at + \alpha$ 为过 $(\alpha, 0)$ 的特征线

沿特征线 $U(t) = u(t, at+\alpha)$

$$\begin{cases} \dfrac{dU(t)}{dt} = \dfrac{\partial u}{\partial x}\cdot a + \dfrac{\partial u}{\partial t} = bU + f \\ U(0) = u(0,\alpha) = \varphi(\alpha) \end{cases}$$

解出 $U(t) = e^{-\int_0^t b(s,x(s))ds}\left(\int_0^t f(s,x(s))\, e^{\int_0^s b(\tau,x(\tau))d\tau} ds + C\right)$

$$u(x,t) = e^{-\int_0^t b(s,x(s))ds}\left(\int_0^t f(s,x(s))\, e^{\int_0^s b(\tau,x(\tau))d\tau} ds + \varphi(x-at)\right)$$

$$= \int_0^t f(s,x(s))\, e^{\int_t^s b(\tau,x(\tau))d\tau} ds + \varphi(x-at)\, e^{-\int_0^t b(s,x(s))ds}$$

例3：$\begin{cases} u_t + a u_x = f(t,x) & t>0,\ x\in\mathbb{R} \\ u|_{t=0} = \varphi(x) \end{cases}$

解：$X(t) = at+\alpha$.　令 $U(t) = u(t, X(t))$

$$\frac{dU(t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dX}{dt} = f(t,x)$$

$$\Rightarrow U(t) = \varphi(\alpha) + \int_0^t f(s, as+\alpha)\,ds$$

$$\Rightarrow u(x,t) = \varphi(x-at) + \int_0^t f(s, as+x-at)\,ds.$$

例4：$\begin{cases} u_t + a(t,x) u_x = 0 \\ u|_{t=0} = \varphi(x) \end{cases}$

解：$\begin{cases} \dfrac{dX}{dt} = a(x,t) \\ X|_{t=0} = \alpha \end{cases}$　为特征线方程

令 $U(t) = u(X(t), t)$　则有

$$\frac{dU}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = a(x,t)u_x + u_t = 0$$

$$\Rightarrow U(t) = U(0) = \varphi(\alpha).$$

设 $\alpha = \alpha(x,t)$.　则有 $u(x,t) = \varphi(\alpha(x,t))$

例5：$\begin{cases} u_t + (x\cos t) u_x = 0 & t>0,\ x\in\mathbb{R} \\ u|_{t=0} = \dfrac{1}{1+x^2} \end{cases}$

解：$\begin{cases} \dfrac{dX}{dt} = x\cos t \\ x(t_0) = x_0 \end{cases} \Rightarrow X = x_0 e^{\sin t - \sin t_0}$

由 $X(t, t_0, x_0) = x_0 e^{\sin t - \sin t_0} \Rightarrow x(t, 0, \alpha) = \alpha e^{\sin t}$

$\Rightarrow \alpha = x e^{-\sin t}$　　再结合下式

$$U(t)=U(0)=u(\alpha.0)=\varphi(\alpha)=\frac{1}{1+\alpha^2}$$

故 $u(x,t)=\dfrac{1}{1+x^2e^{-2\sinh t}}$

下面考虑一般情形 $u_t+a(x,t)u_x=b(x,t)u+f(x,t)$

例6：$\begin{cases} xu_t-tu_x=u & t>0,\ x>0 \\ u(0,x)=g(x) & x>0 \end{cases}$

解：$\begin{cases} \dfrac{dx}{dt}=-\dfrac{t}{x} \\ x(0)=\alpha>0 \end{cases} \Rightarrow x^2+t^2=\alpha^2$

$U(t)\triangleq u(t,x(t,\alpha))$

$$\frac{dU}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{dx}{dt}=u_t-\frac{t}{x}u_x=\frac{U(t)}{x}$$

从而 $U(t)=U(0)e^{\int_0^t\frac{ds}{x(s)}}=g(\alpha)e^{\int_0^t\frac{ds}{\sqrt{\alpha^2-s^2}}}$

$=g(\alpha)e^{\arcsin\frac{t}{\alpha}}$ 即有

$$u(t,x)=g(\sqrt{x^2+t^2})e^{\arcsin\frac{t}{\sqrt{t^2+x^2}}}$$

$=g(\sqrt{x^2+t^2})e^{\arctan\frac{t}{x}}$ 为原方程的解.

§11.2 一阶线性偏微分方程

本节讨论形如 $b_iu_i+cu=f$ 的方程

(1) 齐次方程 $\displaystyle\sum_{i=1}^{n}b_i(x)\frac{\partial u}{\partial x_i}=0$

其特征方程为 $\dfrac{dx_1}{b_1(x)}=\cdots=\dfrac{dx_n}{b_n(x)}$

得到 $n-1$ 个常微分方程 $\dfrac{dx_j}{dx_n}=\dfrac{b_j}{b_n}$ $j=1,2\cdots n-1$

有 $n-1$ 个首次积分，一般地有以下的定理

定理1：若 $\varphi(x_1, x_2 \cdots x_n) = h$ 为特征方程的首次积分

则 $\xi = \varphi(x_1 \cdots x_n)$ 为齐次线性一阶PDE的一个解.

证明：因为 $\varphi(x_1 \cdots x_n) = h$ 为首次积分.

故沿特征线. $\dfrac{dx_i}{dt} = b_i \quad (i=1,2\cdots n)$.

$$0=\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial \varphi}{\partial x_i} b_i$$
从而 φ 是解.

定理2：设 $\varphi_1, \varphi_2 \cdots \varphi_{n-1}$ 为 $n-1$ 个首次积分（独立）

则 $u = g(\varphi_1, \varphi_2 \cdots \varphi_{n-1})$

证明：任取 φ_n 使得其与 $\varphi_1 \cdots \varphi_{n-1}$ 独立.

即 $J(\varphi_1 \cdots \varphi_n) = \dfrac{\partial(\varphi_1 \cdots \varphi_n)}{\partial(x_1 \cdots x_n)}$

$$= \begin{vmatrix} \dfrac{\partial \varphi_1}{\partial x_1} & \cdots & \dfrac{\partial \varphi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \varphi_n}{\partial x_1} & \cdots & \dfrac{\partial \varphi_n}{\partial x_n} \end{vmatrix} \neq 0.$$

记 $\xi_j = \varphi_j(x_1 \cdots x_n) \qquad j = 1,2 \cdots n$.

方程化为以 $(\xi_1, \cdots \xi_n)$ 作为自变量

则 $\displaystyle\sum_j b_j \frac{\partial u}{\partial x_j} = \sum_{j,l} \frac{\partial u}{\partial \xi_l}\frac{\partial \varphi_l}{\partial x_j}\cdot b_j$

$\displaystyle= \sum_l \frac{\partial u}{\partial \xi_l}\left(\sum_j \frac{\partial \varphi_l}{\partial x_j}\cdot b_j\right)$ 当 $l = 1,2 \cdots n-1$ 时.

$\displaystyle\sum_j \frac{\partial \varphi_l}{\partial x_j}\cdot b_j = 0$. 故上式 $\displaystyle= \frac{\partial u}{\partial \xi_n}\left(\sum_j \frac{\partial \varphi_n}{\partial x_j}\cdot b_j\right)$

$\because \varphi_1 \cdots \varphi_{n-1}$ 为独立的首次积分. 则 φ_n 不为方程解.

从而 $\displaystyle\sum_j \frac{\partial \varphi_n}{\partial x_j}\cdot b_j \neq 0$. 故 $\dfrac{\partial u}{\partial \xi_n} = 0$.

即 $u = u(\xi_1 \cdots \xi_{n-1}) = u(\varphi_1 \cdots \varphi_{n-1})$

例1： $\begin{cases} \sqrt{x}\, u_x + \sqrt{y}\, u_y + z u_z = 0 \\ u|_{z=1} = xy \end{cases}$

解： $\dfrac{dx}{\sqrt{x}} \overset{(1)}{=} \dfrac{dy}{\sqrt{y}} \overset{(2)}{=} \dfrac{dz}{z}$

$(1) \Leftrightarrow d(\sqrt{x}-\sqrt{y}) = 0 \qquad (2) \Leftrightarrow d(2\sqrt{y} - \ln z) = 0$

则 $u(x,y,z) = g(\sqrt{x}-\sqrt{y},\ 2\sqrt{y} - \ln z)$

$u|_{z=1} = g(\sqrt{x}-\sqrt{y},\ 2\sqrt{y}) = xy$.

记 $p = \sqrt{x}-\sqrt{y}$，$q = 2\sqrt{y}$，则有

$x = (p+\frac{q}{2})^2 \qquad y = \frac{q^2}{4}$

故 $g(p,q) = (p+\frac{q}{2})^2 \cdot \frac{q^2}{4}$.

从而 $u = \left(\sqrt{x} - \frac{\ln z}{2}\right)^2 \cdot \left(\sqrt{y} - \frac{\ln z}{2}\right)^2$

(2) 非齐次方程 $\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f$.

例2： $\begin{cases} u_t = x u_x + y u_y + u + xy \\ u|_{t=0} = \varphi(x,y) \end{cases}$

解：特征方程 $\dfrac{dt}{b_1} = \dfrac{dx}{b_2} = \dfrac{dy}{b_3}$

即为 $\dfrac{dt}{1} = \dfrac{dx}{-x} = \dfrac{dy}{-y}$

得 $xe^t = c_1$，$ye^t = c_2$.

令 $\xi_1 = xe^t$，$\xi_2 = ye^t$ $\quad \xi_3 = t$ 代入有下式

$\dfrac{\partial u}{\partial \xi_3} = u(\xi_1,\xi_2,\xi_3) + \xi_1 \xi_2 e^{-2\xi_3}$ 可解出 u.

$$u = e^{\xi_3}\left(\int_0^{\xi_3} \xi_1\xi_2 e^{-3\tau}d\tau + g(\xi_1, \xi_2)\right)$$

$$= e^{\xi_3}\left(-\frac{1}{3}\xi_1\xi_2 e^{-3\xi_3} + \frac{1}{3}\xi_1\xi_2 + g(\xi_1, \xi_2)\right)$$

$$= e^{t}\left(-\frac{1}{3}xye^{-t} + \frac{1}{3}xye^{2t} + g(xe^t, ye^t)\right)$$

$$= -\frac{1}{3}xy + e^t\left[\frac{1}{3}xye^{2t} + g(xe^t, ye^t)\right]$$

而 $u\big|_{t=0} = \varphi(x,y) \Rightarrow g = \varphi$.

从而 $u = -\frac{1}{3}xy + e^t\left[\frac{1}{3}xye^{2t} + \varphi(xe^t, ye^t)\right]$

§11.3. Burgers 方程.

本节讨论一阶拟线性偏微分方程.

$$\sum_j b_j \frac{\partial u}{\partial x_j} = c(x,u) \quad \cdots\cdots ①$$

定理1：若 $W = \varphi(x,u)$ 为方程

$$\sum_{j=1}^{n} b_j(x,u)\frac{\partial\varphi}{\partial x_j} + c(x,u)\frac{\partial\varphi}{\partial u} = 0 \text{ 的解}$$

则 $\varphi(x,u) = 0$ 为 ① 的显式解

证明：$\varphi(x_1 \cdots x_n, u(x)) = 0$

求 $\frac{\partial}{\partial x_i}$ 得：$\frac{\partial\varphi}{\partial x_i} + \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x_i} = 0$.

$$\Rightarrow c(x,u) = \sum_{j=1}^{n} b_j(x,u)\frac{\partial u}{\partial x_j}$$

从而 u 为 ① 的解.

以下讨论一类特殊的拟线性方程（Burgers 方程）

$$\begin{cases} u_t + uu_x = 0 \\ u\big|_{t=0} = \varphi(x) \end{cases}$$

定理2： $\begin{cases} u_t + uu_x = 0 \\ u\big|_{t=0} = \varphi(x) \end{cases}$ $\varphi \in C^1(\mathbb{R})$

则其存在 C^1 整体解的充要条件是 $\varphi'(x) \geq 0$.

证明： 特征线 $X(t)$ 满足

$$\frac{dX(t)}{dt} = u(t, X(t)) \triangleq U(t).$$

$$\text{则 } \left.\frac{dU(t)}{dt}\right|_{X(t)} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dX}{dt} = u_t + uu_x = 0.$$

从而沿 $X(t)$. $U(t)$ 为常数. 记 $X(0) = \alpha$.

从而有 $U(t) = U(0) = u(0, X(0)) = \varphi(\alpha)$.

由 $\dfrac{dX}{dt} = U(t) \Rightarrow X(t) = X(0) + t\varphi(\alpha) = \alpha + t\varphi(\alpha)$

下证 $\varphi'(x) \geq 0 \Leftrightarrow \exists!$ 的 $\alpha(t,x)$. 使得

$x = \alpha + t\varphi(\alpha)$.

$(\Rightarrow)$. $\because \varphi'(x) \geq 0$. 故 $\dfrac{dx}{d\alpha} = 1 + t\varphi'(\alpha) > 0$.

由隐函数定理知 $\alpha = \alpha(t,x)$ 存在且唯一.

$(\Leftarrow)$. 若 $\exists x_0$ s.t. $\varphi'(x_0) < 0$. 则 φ 不是单调增的.

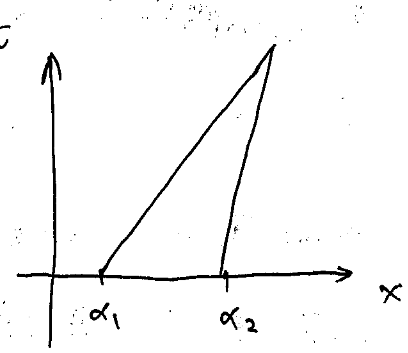
故 $\exists\ \alpha_1 < \alpha_2$ s.t. $\varphi'(\alpha_1) > \varphi(\alpha_2)$

考虑特征线

$X_1(t) = \alpha_1 + t\varphi(\alpha_1)$

$X_2(t) = \alpha_2 + t\varphi(\alpha_2)$

则在 $t = \dfrac{\varphi(\alpha_1) - \varphi(\alpha_2)}{\alpha_2 - \alpha_1}$ 时



两特征线会相交. 而 u 在 $X_1(t)$ 与 $X_2(t)$ 上的值分别为 $\varphi(\alpha_1) > \varphi(\alpha_2)$. 从而在交点处得到矛盾.

由上述推论知 $\varphi'(x) \geq 0 \Leftrightarrow \exists$ 唯一 $\alpha(t,x)$

s.t. $x = \alpha + t\varphi(\alpha)$ 成立 $\Leftrightarrow$ 方程有 C^1 整体解

代入 $u(t) = \varphi(\alpha)$ 知 $u(t,x) = \varphi(\alpha(t,x))$

例1：$\begin{cases} u_t + uu_x = 0 \\ u|_{t=0} = \sin x \end{cases}$

解：$\frac{dx}{d\alpha} = 1 + t\varphi'(\alpha) = 1 + t\cos\alpha$．当 $t \in (0,1)$ 时

$\frac{dx}{d\alpha} > 0$．从而方程在 $t \in (0,1)$ 内有解．

例2：$\begin{cases} u_t + uu_x = 0 \\ u|_{t=0} = \tan x \end{cases}$

解：$\varphi'(x) = \sec^2 x \geq 0$．故方程有整体解．

例3：$\begin{cases} u_t + uu_x = 0 \\ u|_{t=0} = e^{-x^2} \end{cases}$

解：$x(t) = \alpha + \varphi(\alpha)t$．

则 $\frac{dx}{d\alpha} = 1 + t\varphi'(\alpha) = 1 - 2\alpha e^{-\alpha^2} t$．

故 $\frac{dx}{d\alpha} > 0 \Leftrightarrow t < \frac{e^{\alpha^2}}{2\alpha}$．

又 $t > 0$ 知 $\alpha > 0$．记 $f(\alpha) = \frac{e^{\alpha^2}}{2\alpha}$．

则 $f'(\alpha) = e^{\alpha^2} - \frac{e^{\alpha^2}}{2\alpha^2} = 0 \Rightarrow \alpha = \frac{\sqrt{2}}{2}$，

从而 $f(\alpha) \geq f(\frac{\sqrt{2}}{2}) = \frac{e^{\frac{1}{2}}}{\sqrt{2}} \approx 1.16$．

故当 $t \in (0, 1.16)$ 时方程的解

$u(t,x) = \varphi(\alpha(t,x)) = e^{-\alpha^2}$ 存在

定理5：设 $\begin{cases} u_t + a(u)\,u_x = 0 \\ u|_{t=0} = \varphi(x) \end{cases}$

则若 $\frac{d}{dx}\left(a(\varphi(x))\right) \geq 0$. 则方程解整体存在

证明：$\frac{dX(t)}{dt} = a(u(t, X(t))) \triangleq U(t)$. 记 $X(0) = \alpha$.

则 $\left.\frac{dU(t)}{dt}\right|_{X(t)} = a'(u)\,u_t + a'(u)\,u_x \cdot \frac{dX}{dt}$

$= a'(u)\left(u_t + u_x \cdot a(u)\right) = 0$. 从而有

$U(t) = U(0) = a(u(0, X(0))) = a(\varphi(\alpha))$

故 $X(t) = a(\varphi(\alpha))\,t + \alpha$

$\frac{dX}{d\alpha} = 1 + t\,a'(\varphi(\alpha)) \cdot \varphi'(\alpha) > 0$.

从而由隐函数定理知 α 可被 x. t 解出.

于是 $u(t,x) = \varphi(\alpha(t,x))$ 为原方程的解.

注：称方程 $u_t + a(u)\,u_x = 0$ 为守恒律方程.

易见当 $a(u) = u$ 时即为 Burgers 方程

其为最简单的一阶拟线性方程.

第十二章 波动方程

§12.1：一维波动方程的初值问题

$$\begin{cases} u_{tt} = a^2 u_{xx} & x \in \mathbb{R} \quad t \in \mathbb{R}^+ \quad (a>0) \\ u(x,0) = \varphi(x) & u_t(x,0) = \psi(x) \end{cases}$$

d'Alembert 公式（即解的表达式）

解法一：$u_{tt} - a^2 u_{xx} = 0$

$$\Leftrightarrow \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right) u = 0$$

令 $v(x,t) = \left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right) u$

则有 $\begin{cases} v_t + a v_x = 0 \\ v(x,0) = \psi(x) - a\varphi'(x) \end{cases}$

从而由特征线法知 $v(x,t) = \psi(x-at) - a\varphi'(x-at)$

代入有 $u_t - a u_x = \psi(x-at) - a\varphi'(x-at)$

记 $g(x,t) = \psi(x-at) - a\varphi'(x-at)$

则有 $u_t - a u_x = g$.

令 $z(s) = u(x-as, t+s)$，则有

$$z'(s) = -a u_x(x-as, t+s) + u_t(x-as, t+s)$$
$$= g(x-as, t+s)$$

故 $u(x,t) - \varphi(x+at) = \int_{-t}^{0} \frac{dz}{ds} ds$

$$= \int_{-t}^{0} g(x-as, t+s)\, ds$$

从而 $u(x,t) = \varphi(x+at) + \int_0^t g(x-a(s-t), s)\, ds$

$$= \varphi(x+at) + \int_0^t \left[\psi(x-2as+at) - a\varphi'(x-2as+at)\right] ds$$

$$= \varphi(x+at) + \int_{x+at}^{x-at} \left[\psi(y) - a\varphi'(y)\right] \frac{-dy}{2a}$$

$$= \varphi(x+at) + \frac{1}{2a}\int_{x-at}^{x+at}[\psi(y) - a\varphi'(y)]\,dy$$

$$= \frac{1}{2}[\varphi(x+at)+\varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\,dy$$

以下验证上式给出了原方程的一个解

$$u_x = \frac{1}{2}[\varphi'(x+at)+\varphi'(x-at)] + \frac{1}{2a}[\psi(x+at) - \psi(x-at)]$$

$$u_{xx} = \frac{1}{2}[\varphi''(x+at)+\varphi''(x-at)] + \frac{1}{2a}[\psi'(x+at) - \psi'(x-at)]$$

$$u_t = \frac{1}{2}[a\varphi'(x+at) - a\varphi'(x-at)] + \frac{1}{2a}[a\psi(x+at) + a\psi(x-at)]$$

$$u_{tt} = \frac{1}{2}[a^2\varphi''(x+at) + a^2\varphi''(x-at)] + \frac{a}{2}[\psi'(x+at) - \psi'(x-at)]$$

易见 $u_{tt} = a^2 u_{xx}$ 且 $u|_{t=0} = \varphi(x)$. $u_t|_{t=0} = \psi(x)$.

解法二: 令 $\xi = x-at$, $\eta = x+at$.

则有 $u_x = u_\xi + u_\eta \qquad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$.

$u_t = -a u_\xi + a u_\eta \qquad u_{tt} = a^2 u_{\xi\xi} - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta}$.

由 $u_{tt} = a^2 u_{xx} \Rightarrow u_{\xi\eta} = 0$

$\Rightarrow u = f(\xi) + g(\eta) = f(x-at) + g(x+at)$

由 $u|_{t=0} = f(x) + g(x) = \varphi(x)$

及 $u_t|_{t=0} = -af'(x) + ag'(x) = \psi(x)$

$$\Rightarrow \begin{cases} f(x) + g(x) = \varphi(x) \\ a(g(x) - f(x)) = \int \psi(x)\,dx \end{cases}$$

$$\Rightarrow \begin{cases} 2f(x) = \varphi(x) - \frac{1}{a}\int \psi(x)\,dx \\ 2g(x) = \varphi(x) + \frac{1}{a}\int \psi(x)\,dx \end{cases}$$

设 $\int \psi(x)\,dx = \int_0^x \psi(s)\,ds + C$.

则有 $\begin{cases} f(x-at) = \frac{1}{2}\varphi(x-at) - \frac{1}{2a}\int_0^{x-at}\psi(s)ds - \frac{C}{2a} \\ g(x+at) = \frac{1}{2}\varphi(x+at) + \frac{1}{2a}\int_0^{x+at}\psi(s)ds + \frac{C}{2a} \end{cases}$

故 $u(x,t) = f(x-at) + g(x+at)$

$= \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds$.

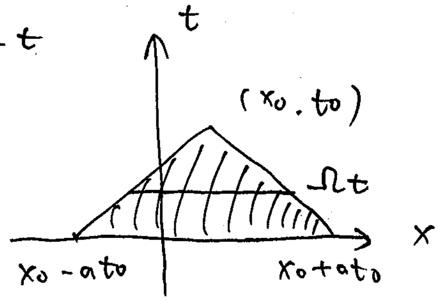
以下证明解的唯一性.

设方程有两个解 u_1, u_2. 定义 $w \triangleq u_1 - u_2$.

则有 $\begin{cases} w_{tt} = a^2 w_{xx} \\ w|_{t=0} = 0, \quad w_t|_{t=0} = 0 \end{cases}$.

只要证明 $w \equiv 0$ 即可. 在 (x_0, t_0) 处.

由 d'Alembert 公式知 $w(x_0, t_0)$ 只由 $[x_0 - at_0, x_0 + at_0]$ 上的取值决定. 称其为依赖区域 Ω_t



定义 $E(t) = \frac{1}{2}\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)}(w_t^2 + a^2 w_x^2)\,dx$

则有 $E(0) = \frac{1}{2}\int_{x_0-at_0}^{x_0+at_0}(w_t^2 + a^2 w_x^2)\Big|_{t=0}dx = 0$.

从而又由 $\frac{dE(t)}{dt} = -\frac{a}{2}\underbrace{(w_t^2 + a^2 w_x^2)\Big|_{x_0+a(t_0+t)}}_{①}$

$- \frac{a}{2}\underbrace{(w_t^2 + a^2 w_x^2)\Big|_{x_0-a(t_0-t)}}_{②} + \int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)}(w_t w_{tt} + a^2 w_x w_{xt})\,dx$

$= ① + ② + \int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)}(w_t w_{tt} + a^2(w_t w_x)_x - a^2 w_t w_{xx})\,dx$

$= ① + ② + a^2 (w_t w_x)\Big|_{x_0-a(t_0-t)}^{x_0+a(t_0-t)}$

$= -\frac{a}{2}[w_t^2 + a^2 w_x^2 - 2a w_t w_x]\Big|_{x_0+a(t_0-t)}$

$-\frac{a}{2}[w_t^2 + a^2 w_x^2 + 2a w_t w_x]\Big|_{x_0-a(t_0-t)} \leq 0$

从而可知 $E(t)$ 在 $[0, t_0]$ 上为减函数

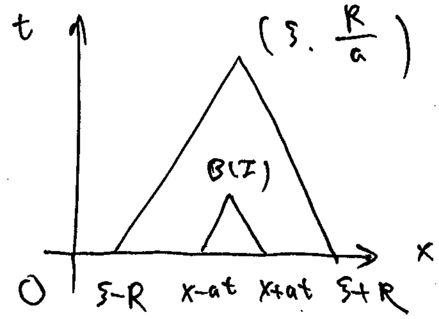
由表达式知 $E(t) \geq 0$. 但 $E(0) = 0$. 故有 $E(t) \equiv 0$

从而有 $w_t \equiv 0$. $w_x \equiv 0$ in Ω_t $\quad t \in (0, t_0)$

故有 $w \equiv const$. 结合 $w\big|_{t=0} = 0$ 知

$w \equiv 0$. 即 $u_1 \equiv u_2$. 从而证明了解唯一.

决定区域:



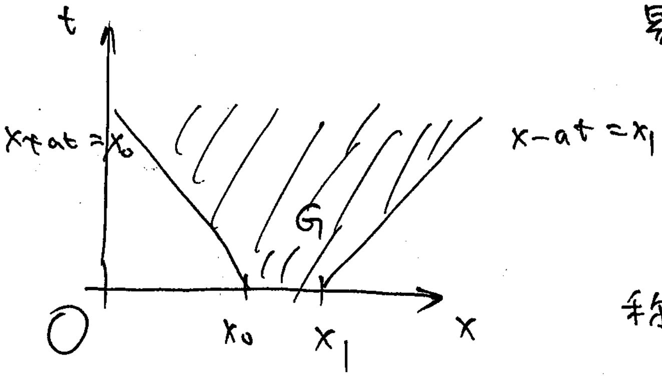
易见 $\forall (x, t) \in B(I)$

$u(x, t)$ 的值由

$[\xi - R, \xi + R]$ 上的值决定.

称三角形 $B(I)$ 为其决定区域

影响区域:



易见 $[x_0, x_1]$ 内的初值

会影响到 G 内的 $u(x, t)$

的值. 而不会影响 G 之

外的 $u(x, t)$.

称 G 为 $[x_0, x_1]$ 的影响区域.

下面讨论半直线 $R_+ = \{x > 0\}$ 上的初边值问题.

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & x \in \mathbb{R}_+ \quad t > 0 \\ u(x, 0) = g & u_t(x, 0) = h \quad x \in \overline{R_+} \\ u(0, t) = 0 & (t \geq 0) \end{cases}$$

其中 g, h 为已知函数. 满足 $g(0) = h(0) = 0$.

解: 先把问题转化到全空间 $\mathbb{R}$ 上

为此对 u, g, h 作奇性延拓

即定义 $\bar{u}(x,t)=\begin{cases} u(x,t) & x\geq 0,\ t\geq 0 \\ -u(-x,t) & x\leq 0 \quad t\geq 0 \end{cases}$

$\bar{g}(x)=\begin{cases} g(x) & x\geq 0 \\ -g(-x) & x\leq 0 \end{cases}$ $\quad \bar{h}(x)=\begin{cases} h(x) & x\geq 0 \\ -h(-x) & x\leq 0 \end{cases}$

则有 $\begin{cases} \bar{u}_{tt}-a^2\bar{u}_{xx}=0 & (x,t)\in\mathbb{R}\times(0,+\infty) \\ \bar{u}(x,0)=\bar{g}(x) \quad \bar{u}_t(x,0)=\bar{h}(x) & x\in\mathbb{R}. \end{cases}$

由 d'Alembert 公式有

$$\bar{u}(x,t)=\frac{1}{2}\left[\bar{g}(x+at)+\bar{g}(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\bar{h}(y)\,dy$$

则当 $x\geq at\geq 0$ 时有

$$u(x,t)=\frac{1}{2}\left[g(x+at)+g(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}h(y)\,dy$$

而当 $0\leq x\leq at$ 时有

$$u(x,t)=\frac{1}{2}\left[g(x+at)-g(at-x)\right]+\frac{1}{2a}\int_{0}^{x+at}h(y)\,dy$$
$$+\frac{1}{2a}\int_{x-at}^{0}-h(-y)\,dy$$
$$=\frac{1}{2}\left[g(x+at)-g(at-x)\right]+\frac{1}{2a}\int_{at-x}^{x+at}h(y)\,dy.$$

以下讨论非齐次方程

$$\begin{cases} u_{tt}-a^2u_{xx}=f(x,t) \\ u|_{t=0}=\varphi(x) \\ u_t|_{t=0}=\psi(x) \end{cases}$$

首先考虑方程 $\begin{cases} w_{tt}-a^2w_{xx}=h(x,t) \\ w|_{t=0}=0 \\ w_t|_{t=0}=0 \end{cases}$

从考虑如下的初值问题开始

$$\begin{cases} v_{tt} = a^2 v_{xx} & x\in\mathbb{R} \quad t \geq s \\ v(x,s;s) = 0, & v_t(x,s;s) = h(x,s) \end{cases}$$

可记 $v(x,t;s) = \tilde{v}(x,t-s,s)$.

则有 $$\begin{cases} \tilde{v}_{tt} = a^2 \tilde{v}_{xx} & x\in\mathbb{R}, \quad t\geq 0 \\ \tilde{v}(x,0;s) = 0 & \tilde{v}_t(x,0;s) = h(x,s) \end{cases}$$

由 d'Alembert公式有

$$\tilde{v}(x,t;s) = \frac{1}{2a}\int_{x-at}^{x+at} h(r,s)\,dr$$

从而 $$v(x,t;s) = \frac{1}{2a}\int_{x-a(t-s)}^{x+a(t-s)} h(r,s)\,dr$$

Duhamel 原理：如下初值问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = h(x,t) \\ u|_{t=0} = 0 \\ u_t|_{t=0} = 0 \end{cases}$$

的解为 $$u(x,t) = \frac{1}{2a}\int_0^t \int_{x-a(t-s)}^{x+a(t-s)} h(r,s)\,dr\,ds$$

证明：由 $u(x,t) = \int_0^t \tilde{v}(x,t-s;s)\,ds$

知 $$u_t(x,t) = \tilde{v}(x,0;t) + \int_0^t \tilde{v}_t(x,t-s;s)\,ds$$

$$u_{tt} = \tilde{v}_t(x,0;t) + \int_0^t \tilde{v}_{tt}(x,t-s;s)\,ds$$

$$= h(x,t) + \int_0^t a^2 \tilde{v}_{xx}(x,t-s;s)\,ds$$

$$= h(x,t) + a^2 u_{xx}(x,t).$$

从而 $u(x,t)$ 满足 $u_{tt} - a^2 u_{xx} = h(x,t)$

易见 $u|_{t=0} = 0$. $u_t|_{t=0} = 0$ 故结论得证.

例1：$\begin{cases} u_{tt}-u_{xx}=x-t & (x,t)\in\mathbb{R}\times(0,+\infty) \\ u(x,0)=x^2 \quad u_t(x,0)=\sin x \end{cases}$

解：分为两个方程考虑．易见 $u=v+w$．

$$\begin{cases} v_{tt}-v_{xx}=0 \\ v|_{t=0}=x^2 \\ v_t|_{t=0}=\sin x \end{cases} \quad 与 \quad \begin{cases} w_{tt}-w_{xx}=x-t \\ w|_{t=0}=0 \\ w_t|_{t=0}=0 \end{cases}$$

易见（从 d'Alembert 公式可得）

$$v(x,t)=\frac{1}{2}\left[(x+t)^2+(x-t)^2\right]+\frac{1}{2}\int_{x-t}^{x+t}\sin y\,dy$$

$$=x^2+t^2+\frac{1}{2}\left[\cos(x-t)-\cos(x+t)\right]$$

由 Duhamel 原理知

$$w(x,t)=\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)}(r-s)\,dr\,ds$$

$$=\frac{1}{2}\int_0^t\left(\frac{r^2}{2}-rs\right)\Big|_{x-t+s}^{x+t-s}ds$$

$$=\frac{1}{2}\int_0^t\left[\frac{(x+t-s)^2}{2}-\frac{(x+s-t)^2}{2}-s(x+t-s)+s(x+s-t)\right]ds$$

$$=\frac{1}{2}\int_0^t\left(2s^2-2s(x+t)+\frac{(x+t)^2}{2}-\frac{(x-t)^2}{2}\right)ds$$

$$=\frac{t^3}{3}-\frac{t^2(x+t)}{2}+t^2x=-\frac{t^3}{6}+\frac{t^2x}{2}$$

从而有 $u(x,t)=v(x,t)+w(x,t)$

$$=x^2+t^2+\frac{1}{2}\left[\cos(x-t)-\cos(x+t)\right]-\frac{t^3}{6}+\frac{t^2x}{2}$$

$$=x^2+t^2-\frac{t^3}{6}+\frac{t^2x}{2}+\sin x\sin t$$

易于验证上式为原方程的解

波方程的唯一性定理（有限区间情形）

$$\begin{cases} u_{tt}-a^2u_{xx}=f(x,t) & x\in[0,l],\quad t\geq 0 \\ u|_{t=0}=\varphi(x) \quad u_t|_{t=0}=\psi(x) & \text{解是唯一的.} \\ u|_{x=0}=g(t) \quad u|_{x=l}=h(t) \end{cases}$$

证明：设 u_1, u_2 为原方程的两个解.

定义 $w\triangleq u_1-u_2$. 则 w 满足

$$\begin{cases} w_{tt}-a^2w_{xx}=0 & x\in[0,l] \quad t\geq 0 \\ w|_{t=0}=0 & w_t|_{t=0}=0 \\ w|_{x=0}=0 & w|_{x=l}=0 \end{cases}$$

定义 $E(t)=\int_0^l (w_t^2+a^2w_x^2)\,dx$

则有 $E(0)=0$. 且有

$$\frac{dE(t)}{dt}=\int_0^l(2w_tw_{tt}+2a^2w_xw_{xt})\,dx$$

$$=2\int_0^l(w_tw_{tt}+a^2(w_xw_t)_x-a^2w_{xx}w_t)\,dx$$

$$=2\int_0^l w_t(w_{tt}-a^2w_{xx})\,dx+2a^2w_xw_t\Big|_0^l$$

$$=2a^2w_xw_t\Big|_0^l$$

由 $w|_{x=0}=0 \Rightarrow w_t|_{x=0}=0$.

同理 $w|_{x=l}=0 \Rightarrow w_t|_{x=l}=0$.

故 $E'(t)\equiv 0 \Rightarrow E(t)\equiv 0$.

从而有 $w_t\equiv 0$. $w_x\equiv 0 \Rightarrow w\equiv \text{const}$

结合 w 在边界处取零值知 $w\equiv 0$

从而 $u_1\equiv u_2$. 这就证明了唯一性

下面求解有限区间上的波动方程

例1：
$$\begin{cases} u_{tt} = 4u_{xx} & x\in[0,1] \\ u|_{t=0} = 4\sin^3(\pi x) & u|_{x=0}=u|_{x=1}=0 \\ u_t|_{t=0} = 30x(1-x) \end{cases}$$

Step 1：求 $E(1)$，其中 $E(t)=\frac{1}{2}\int_0^1 (u_t^2+4u_x^2)\,dx$

$$\frac{d}{dt}E(t) = \int_0^1 (u_t u_{tt} + 4u_x u_{xt})\,dx$$

$$= 4u_x u_t\Big|_0^1 + \int_0^1 (u_t u_{tt} - 4u_t u_{xx})\,dx$$

$$= 4u_t u_x\Big|_0^1 = 0$$

从而 $E(1) = E(0) = \frac{1}{2}\int_0^1 900x^2(1-x)^2\,dx$

$$+ 288\pi^2\int_0^1 \sin^4(\pi x)\cos^2(\pi x)\,dx = 15+18\pi^2$$

Step 2：求解方程，并求 $u(x,2)$.

设 $u(x,t) = T(t)X(x)$.

则 $T''X = 4TX'' \Rightarrow \frac{T''}{4T} = \frac{X''}{X} = -\lambda$.

$$\Rightarrow X = c_1\sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x)$$

由 $X(0)=X(1)=0$ 知 $\lambda=(n\pi)^2 \quad n\in N$.

且 $X = c_1\sin(n\pi x)$ ，从而有

$$u_n(x,t) = [A_n\sin(2n\pi t) + B_n\cos(2n\pi t)]\sin(n\pi x)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t).$$

$u|_{t=0} = \sum_{n=1}^{\infty} B_n\sin(n\pi x) = 4\sin^3(\pi x)$ 由三倍角公式

$\Rightarrow B_1=3,\quad B_3=-1\quad$ 其余 $B_n=0$.

$$u_t\big|_{t=0} = \sum_{n=1}^{\infty} 2n\pi A_n \sin(n\pi x) = 30x(1-x)$$

$$2n\pi A_n = 2\int_0^1 30x(1-x)\sin(n\pi x)\,dx$$

$$\Rightarrow A_n = \frac{1}{n\pi}\int_0^1 30x(1-x)\sin(n\pi x)\,dx$$

$$u(x,2) = u(x,0) = 4\sin^3(\pi x).$$

Step 3. 收敛性问题.

利用 Weierstrass 判别法. 只要证 $\sum |A_n| + |B_n|$ 收敛

$\sum |B_n|$ 显然收敛. 记 $f(x) = 30x(1-x)$

$$A_n = \frac{1}{n^2\pi^2}\left[f(x)\cos(n\pi x)\Big|_0^1 - \int_0^1 \cos(n\pi x) f'(x)\,dx\right]$$

$$= -\frac{1}{n^2\pi^2}\int_0^1 \cos(n\pi x)(30-60x)\,dx$$

$$|A_n| \leq \frac{C_0}{n^2\pi^2} \quad (C_0 = 60).$$

从而 $\sum_n |A_n|$ 收敛. 故 u 的定义合理

$$又\ A_n = -\frac{1}{n^3\pi^3}\int_0^1 f'(x)\,d(\sin(n\pi x))$$

$$= \frac{1}{n^3\pi^3}\int_0^1 f''(x)\sin(n\pi x)\,dx$$

$$= -\frac{1}{n^4\pi^4}\int_0^1 f''(x)\,d(\cos(n\pi x))$$

$$= \frac{1}{n^4\pi^4} f''(x)(\cos n\pi x)\Big|_0^1 = \frac{-60}{n^4\pi^4}\left[(-1)^n - 1\right]$$

有 $|A_n| \leq \dfrac{C_1}{n^4\pi^4}$ 从而 u_{xx}. u_{tt} 定义也合理

由唯一性知 $u = \sum_{n=1}^{\infty} u_n(x,t)$ 为解.

例2：$\begin{cases} u_{tt} = 4u_{xx} + (x-t) & x\in[0,1] \\ u|_{t=0} = 0 \\ u_t|_{t=0} = 0 \end{cases}$

解：设 $u_n(x,t) = T_n(t)\sin(n\pi x)$

代入方程有 $\sum_{n=1}^{\infty}[T_n''(t) + 4n^2\pi^2 T_n(t)]\sin(n\pi x) = x-t$

$$\begin{cases} u|_{t=0} = \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x) = 0 \\ u_t|_{t=0} = \sum_{n=1}^{\infty} T_n'(0)\sin(n\pi x) = 0 \end{cases}$$

$\Rightarrow \begin{cases} T_n(0) = 0 \\ T_n'(0) = 0 \end{cases}$ 记 $f_n(t) = 2\int_0^1 (x-t)\sin(n\pi x)\,dx$.

则有 $T_n'' + 4n^2\pi^2 T_n = f_n(t)$

$$T_n(t) = c_1\varphi_1 + c_2\varphi_2 + \int_0^t \frac{\varphi_1(s)\varphi_2(t) - \varphi_1(t)\varphi_2(s)}{\varphi_1(s)\varphi_2'(s) - \varphi_2(s)\varphi_1'(s)} f_n(s)\,ds$$

由 φ_1, φ_2 为 $T'' + 4n^2\pi^2 T = 0$ 的两个解.

代入有（令 $c_1 = c_2 = 0$）

$$T_n(t) = \frac{1}{2n\pi}\int_0^t f_n(s)\sin(2n\pi(t-s))\,ds.$$

故 $u(x,t) = \sum_{n=1}^{\infty}\frac{1}{2n\pi}\int_0^t f_n(s)\sin(2n\pi(t-s))\,ds\cdot\sin(n\pi x)$

例3：$\begin{cases} u_{tt} = 4u_{xx} & x\in[0,1] \\ u|_{t=0} = u_t|_{t=0} = 0 \\ u(0,t) = g(t), & u(1,t) = h(t) \end{cases}$

解：设 $V(x,t) = g(t) + x[h(t) - g(t)]$

则有 $V_{xx} = 0$. $V_{tt} = g''(t) + x[h''(t) - g''(t)]$.

令 $W(x,t) = u(x,t) - V(x,t)$

则有 $w_{tt} = u_{tt} - V_{tt}$，$w_{xx} = u_{xx} - V_{xx}$

$$w_{tt} - 4w_{xx} = (u_{tt} - V_{tt}) - 4(u_{xx} - V_{xx})$$

$$= 4V_{xx} - V_{tt} = -g''(t) + x[g''(t) - h''(t)]$$

可转化为之前讨论的情形（例2情形）

$$\because \ w|_{x=0} = u|_{x=0} - V|_{x=0} = 0.$$

$$w|_{x=1} = u|_{x=1} - V|_{x=1} = 0.$$

$$w|_{t=0} = u|_{t=0} - V|_{t=0} = -g(0) + x(g(0) - h(0))$$

$$w_t|_{t=0} = u_t|_{t=0} - V_t|_{t=0} = -g'(0) + x(g'(0) - h'(0))$$

$$\therefore \begin{cases} w_{tt} - 4w_{xx} = -g''(t) + x[g''(t) - h''(t)] \\ w|_{t=0} = -g(0) + x(g(0) - h(0)) \\ w_t|_{t=0} = -g'(0) + x(g'(0) - h'(0)) \\ w(0,t) = w(1,t) = 0. \end{cases}$$

解出 w 即可有 $u = w + V$ 的表达式.

§12-2 调和函数的分离变量法

$$\begin{cases} \Delta u = 0 & \text{in } B_1(0) \subset \mathbb{R}^2 \\ u = f(x,y) & \text{on } \partial B_1(0). \end{cases}$$

设 $u = R(r)\Theta(\theta)$，则有（由第七章）

$$\begin{cases} r^2R'' + rR' = \lambda R \\ \Theta'' = -\lambda\Theta \end{cases}$$

易见 $\lambda > 0$. 设 $\lambda = w^2$. 则有

$\Theta(\theta) = c_1 \cos w\theta + c_2 \sin w\theta$

由 $\Theta(\theta+2\pi) = \Theta(\theta)$ 知 $w = n$, $(n \in N)$.

考虑方程 $r^2 R'' + rR' - n^2 R = 0$.

令 $r = e^s$ 有 $R'(r) = \frac{1}{r}\frac{\partial R}{\partial s}$, $R''(r) = \frac{1}{r^2}\left(\frac{\partial^2 R}{\partial s^2} - \frac{\partial R}{\partial s}\right)$.

从而有 $\frac{\partial^2 R}{\partial s^2} = n^2 R \Rightarrow R = c_3 e^{ns} + c_4 e^{-ns}$.

即 $R = c_3 r^n + c_4 r^{-n}$.

记 $u_0(r,\theta) = c_0 + c_1 \ln r$. 而当 $n \geq 1$ 时, 则定义

$u_n(r,\theta) = (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n})$

结合其在原点有定义的要求.

可知 $u = c_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$. 令 $r=1$.

记 $c_0 = \frac{a_0}{2}$, 由 Fourier 级数知

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$. 故有

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,d\varphi + \frac{1}{\pi}\int_0^{2\pi} r^n f(\varphi)[\cos(n\theta - n\varphi)]\,d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\left[1 + 2\sum_{n=1}^{\infty} r^n \cos n(\varphi-\theta)\right]d\varphi$$

令 $z = re^{i(\theta-\varphi)}$ 则 $z^n = r^n \cos(n(\theta-\varphi)) + i r^n \sin(n(\theta-\varphi))$

故 $1 + 2\sum_{n=1}^{\infty} r^n \cos n(\varphi-\theta)\,d\varphi$

$$= 1 + 2\sum_{n=1}^{\infty} \mathrm{Re}\, z^n = 2\,\mathrm{Re}\frac{1}{1-z} - 1 = \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}$$

从而 $u(r,\theta)=\dfrac{1}{2\pi}\int_0^{2\pi} f(\varphi)\dfrac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}d\varphi$

此即 Poisson 公式.

例 1. $\begin{cases}\Delta u=0 & \text{in } B_1(0)\\ u=\cos\theta & \text{on } \partial B_1(0)\end{cases}$ 求 $u(0,0)$.

解: $u(0,0)=\dfrac{1}{2\pi}\int_0^{2\pi}\cos\varphi\, d\varphi=0$.

例 2. $\begin{cases}\Delta u=12(x^2-y^2) & \text{in } \Omega=\{x \mid a<|x|<b\}\\ u\big|_{r=a}=1 \quad \dfrac{\partial u}{\partial n}\Big|_{r=b}=0\end{cases}$.

唯一性: 设 $v=u_1-u_2$. u_1, u_2 为上述方程解.

则 $\begin{cases}\Delta v=0\\ v\big|_{r=a}=1 \quad \dfrac{\partial v}{\partial n}\Big|_{r=b}=0\end{cases}$.

$$0=\int_\Omega v\Delta v=\int_\Omega (vv_i)_i-|\nabla v|^2\, dx$$

$$=\int_{\partial\Omega} vv_i\cdot\vec{n}_i\, d\sigma-\int_\Omega|\nabla v|^2 dx$$

$$\Rightarrow |\nabla v|\equiv 0 \Rightarrow v\equiv 0 \Rightarrow u_1\equiv u_2.$$

解方程: $\begin{cases}\Delta u=12r^2\cos 2\theta\\ u\big|_{r=a}=1\\ \dfrac{\partial u}{\partial r}\Big|_{r=b}=0\end{cases}$.

设 $u(r,\theta)=A_0+B_0\ln r+(C_2r^2+\tilde{C}_2r^{-2})\cos 2\theta$.

代入有 $\begin{cases}A_0+B_0\ln a=1\\ \dfrac{B_0}{b}=0\end{cases}$

且 $\begin{cases} a^2 C_2 + a^{-2}\tilde{C}_2 = -a^4 \\ 2bC_2 - 2b^{-3}\tilde{C}_2 = -4b^3 \end{cases}$.

$$\Rightarrow \begin{cases} A_0 = 1 \\ B_0 = 0 \end{cases} \qquad C_2 = -\frac{a^6+2b^6}{a^4+b^4} \qquad D_2 = \frac{a^4b^4(2b^2-a^2)}{a^4+b^4}$$

从而 $u(r,\theta) = 1 - \left[\frac{a^6+2b^6}{a^4+b^4} r^2 + \frac{a^4b^4(a^2-2b^2)}{a^4+b^4} r^{-2} - r^4\right] \cos 2\theta$

平均值公式.

由 Poisson 公式知，在 $r=0$ 时有

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\, d\varphi .$$

一般的，$\begin{cases} \Delta u = 0 & \text{in } B_R(x) \subset \mathbb{R}^2 \\ u = f(\theta) & \text{on } \partial B_R(x) \end{cases}$.

有 $u(x) = \frac{1}{2\pi R}\int_0^{2\pi} u(x+R\theta)\, d\theta$.

以下讨论 n 维的平均值公式.

由 $\int_{B_\rho(x)} \Delta u(y)\, dy = \int_{\partial B_\rho(x)} \vec{\nabla} u(y)\cdot \vec{\nu}\, d\sigma_y$

$= \int_{\partial B_\rho(x)} \frac{\partial u}{\partial \rho}(x+\rho w)\, d\sigma_y$ （由 $y = x+\rho w$ 有下式）

$= \rho^{n-1}\int_{\partial B_1(0)} \frac{\partial u}{\partial \rho}(x+\rho w)\, d w_{S^{n-1}}$

$= \rho^{n-1}\frac{\partial}{\partial \rho}\left[\int_{\partial B_1(0)} u(x+\rho w)\, d w_{S^{n-1}}\right]$.

从而若 $\Delta u = 0$，则 $\int_{\partial B_1(0)} u(x+\rho w)\, dw$ 为常数.

而 $\lim\limits_{\rho\to 0}\int_{\partial B_1(0)} u(x+\rho w)\, dw = n\,\omega_n\, u(x)$.

从而 $u(x) = \dfrac{1}{n\omega_n} \displaystyle\int_{\partial B_1(0)} u(x+\rho w)\, d\omega_{S^{n-1}}$

$$= \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho(x)} u(y)\, d\sigma_y$$

其中 $\omega_n = |B_1(0)|$. 即 n 维球的体积.

又由 $\omega_n \rho^{n-1} u(x) = \displaystyle\int_{\partial B_\rho(x)} u(y)\, d\sigma_y$ 知

$$n\omega_n u(x) \int_0^R \rho^{n-1}\, d\rho = \int_0^R \int_{\partial B_\rho(x)} u(y)\, d\sigma_y$$

$$\Rightarrow \omega_n u(x) \cdot R^n = \int_{B_R(x)} u(y)\, dy$$

$$\Rightarrow u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} u(y)\, dy .$$

即平均值公式可以写成上面的形式.

综上所述, 我们有: 若 $\Delta u = 0$ in $B_R(x)$

则 $u(x) = \dfrac{1}{\omega_n R^n} \displaystyle\int_{B_R(x)} u(y)\, dy$

且 $\forall\, 0 < \rho < R$, 有

$$u(x) = \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho(x)} u(y)\, dy .$$

上述两公式统称为 n 维的平均值公式.

注: 事实上可以证明满足平均值公式的函数 u 一定为调和函数. 即上述定理逆命题也成立

§12.3 高维波动方程

Kirchhoff 公式

考虑 $\begin{cases} u_{tt} = a^2 \Delta u & x\in\mathbb{R}^3 \quad t>0 \\ u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$

先找径向解. 设 $\varphi(x)=\varphi(r)$, $\psi(x)=\psi(r)$

又 $\Delta u = u_{rr} + \frac{2}{r}u_r$. 方程化为

$$u_{tt} = a^2\left(u_{rr}+\frac{2}{r}u_r\right)$$

$\Rightarrow (ru)_{tt} = a^2 (ru)_{rr}$ 令 $w = ru$.

则有 $\begin{cases} w_{tt} = a^2 w_{rr}, & r>0 \quad t>0 \\ w|_{t=0} = r\varphi(r) & \\ w_t|_{t=0} = r\psi(r) & r\geq 0 \end{cases}$

将 φ、ψ 奇性延拓到 $(-\infty,0)$

由 d'Alembert 公式有

$$u(r,t) = \begin{cases} \frac{1}{2r}\left[(r+at)\varphi(r+at)+(r-at)\varphi(r-at)\right] \\ \quad + \frac{1}{2a}\int_{r-at}^{r+at} \tau\psi(\tau)\,d\tau & \text{当 } r\geq at\geq 0 \text{ 时} \\ \frac{1}{2r}\left[(r+at)\varphi(r+at)+(r-at)\varphi(at-r)\right] \\ \quad + \frac{1}{2ar}\int_{-r+at}^{r+at} \tau\psi(\tau)\,d\tau & \text{当 } 0\leq r\leq at \text{ 时} \end{cases}$$

下面用球面平均法来推导 Kirchhoff 公式.

$\forall x\in\mathbb{R}^3$, $S_r \triangleq \{y \mid |y-x|=r\}$.

$$\hat{M}u \triangleq \frac{1}{4\pi r^2}\int_{S_r} u(y,t)\,dS_y$$

令 $\omega=\dfrac{y-x}{r}$ 则 $|\omega|=1$. 因有

$$\widehat{M}u=\frac{1}{4\pi}\int_{|\omega|=1}u(x+\omega r,t)\,dS_\omega$$

由 $u_{tt}=a^2\Delta u(y,t)$ 知

$$\int_{|y-x|\le r}u_{tt}\,dy=\int_{|y-x|\le r}a^2\Delta u(y,t)\,dy$$

$$=a^2\int_{S_r}\frac{\partial u}{\partial \omega}(y,t)\,dS_y$$

$$=a^2\sum_{i=1}^{3}\int_{S_r}\omega_i\,u_{y_i}(y,t)\,dS_y$$

$$=a^2r^2\sum_{i=1}^{3}\int_{|\omega|=1}\omega_i u_{y_i}(x+\omega r,t)\,dS_\omega$$

$$=a^2r^2\int_{|\omega|=1}\frac{\partial}{\partial r}u(x+\omega r,t)\,dS_\omega$$

$$=4\pi a^2r^2\frac{\partial}{\partial r}(\widehat{M}u)$$

从而 $4\pi a^2r^2(\widehat{M}u)_r=\int_{|y-x|\le r}u_{tt}\,dy$

$$=\int_0^r d\rho\int_{S_\rho}u_{tt}(y,t)\,dS_y$$

$$\Rightarrow\quad 4\pi a^2\left(r^2(\widehat{M}u)_r\right)_r=\int_{S_r}u_{tt}(y,t)\,dS_y$$

$$=r^2\int_{|\omega|=1}u(x+\omega r,t)\,dS_\omega$$

$$=4\pi\left(r^2\widehat{M}u\right)_{tt}\quad\cdots\ (*)$$

结合 $\left(r^2(\widehat{M}u)_r\right)_r=2r(\widehat{M}u)_r+r^2(\widehat{M}u)_{rr}$

及 $(r\widehat{M}u)_{rr}=2(\widehat{M}u)_r+r(\widehat{M}u)_{rr}$

知 $\left(r^2(\widehat{M}u)_r\right)_r=r\,(r\widehat{M}u)_{rr}$

在(*)式两边同除以 $4\pi r$，则有

$$(r\widehat{M}u)_{tt} = a^2(r\widehat{M}u)_{rr}$$

故 $r\widehat{M}u = w_1(r+at) + w_2(r-at)$

令 $r\to 0$ 有 $w_1(at)+w_2(-at)=0$

$\Rightarrow w_2(x) = -w_1(-x)$，从而 $\forall x\in\mathbb{R}$

$$r\widehat{M}u = w_1(r+at) - w_1(at-r) \quad \cdots (**)$$

$$\Rightarrow \widehat{M}u + r(\widehat{M}u)_r = w_1'(r+at) + w_1'(at-r)$$

令 $r\to 0$ 有 $\lim\limits_{r\to 0}\widehat{M}u = 2w_1'(at)$

而另一方面 $\lim\limits_{r\to 0}\widehat{M}u = \dfrac{1}{4\pi}\displaystyle\int_{|w|=1} u(x,t)\,dS_w = u(x,t)$

从而有 $u(x,t) = 2w_1'(at)$.

下面求出 $w_1'(at)$ 的值，从而得到 $u(x,t)$ 的表达式

由(**)式分别关于 r、t 求偏导数，得

$$\begin{cases} (r\widehat{M}u)_r = w_1'(r+at) + w_1'(at-r) \\ \dfrac{1}{a}(r\widehat{M}u)_t = w_1'(r+at) - w_1'(at-r) \end{cases}$$

再令 $t\to 0$，有

$$\begin{cases} (r\widehat{M}u)_r\big|_{t=0} = w_1'(r) + w_1'(-r) \\ \dfrac{1}{a}(r\widehat{M}u)_t\big|_{t=0} = w_1'(r) - w_1'(-r) \end{cases}$$

相加得 $\underset{①}{(r\widehat{M}u)_r\big|_{t=0}} + \underset{②}{\dfrac{1}{a}(r\widehat{M}u)_t\big|_{t=0}} = 2w_1'(r)$

下面分别计算左边的两项

① $(r\hat{M}u)_r\Big|_{t=0} = \dfrac{1}{4\pi r}\displaystyle\int_{S_r} u(y,t)\,dS_y\Big|_{t=0}$

$= \dfrac{r^2}{4\pi r}\displaystyle\int_{|w|=1} u(x+rw,t)\,dS_w\Big|_{t=0}$

$= \dfrac{r}{4\pi}\displaystyle\int_{|w|=1} \varphi(x+wr)\,dS_w = (r\hat{m}\varphi)_r$

② $\dfrac{1}{a}(r\hat{M}u)_t\Big|_{t=0} = \dfrac{r}{4\pi a}\displaystyle\int_{|w|=1} \psi(x+wr)\,dS_w = \dfrac{r}{a}\hat{m}\psi$

故 $2w_1'(r) = (r\hat{m}\varphi)_r + \dfrac{r}{a}\hat{m}\psi$.

令 $r=at$，则有 $2w_1'(at) = (t\hat{m}\varphi)_t + t\hat{m}\psi$

$= \left(\dfrac{1}{4\pi a^2 t}\displaystyle\int_{S_{at}} \varphi(y)\,dS_y\right)_t + \dfrac{1}{4\pi a^2 t}\displaystyle\int_{S_{at}} \psi(y)\,dS_y$

上式称为 Kirchhoff 公式 .

在球面坐标系下 (即 $\vec{y} = (at, \theta, \phi)$)

若记 $\alpha = \sin\theta\cos\phi$, $\beta = \sin\theta\sin\phi$, $\gamma = \cos\theta$

又 $dS = \sin\theta\, d\theta\, d\phi$, $\theta\in[0,\pi]$, $\phi\in[0,2\pi]$

则由 $y_1 = x_1+\alpha at$, $y_2 = x_2+\beta at$, $y_3 = x_3+\gamma at$

知 Kirchhoff 公式可改写为

$$u(x,t) = \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} \psi(x_1+\alpha at, x_2+\beta at, x_3+\gamma at)\,dS$$

$$+\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} \varphi(x_1+\alpha at, x_2+\beta at, x_3+\gamma at)\,dS\right)$$

注：Kirchhoff 公式可解释声波的传播现象

降维法，Poisson 公式.

（下面由 Kirchhoff 公式得到 Poisson 公式）.

考虑. $$\begin{cases} u_{tt} - a^2 (u_{x_1x_1} + u_{x_2x_2}) = 0 \\ u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$$

定义 $S_{at} \triangleq \{ y \in \mathbb{R}^3 \mid |y-x| = at \}$

$S_{at}^{+} \triangleq S_{at} \cap \{ y_3 > 0 \}$.

则 $y_3 = \sqrt{(at)^2 - (y_1-x_1)^2 - (y_2-x_2)^2} + x_3$.

$$\frac{\partial y_3}{\partial y_i} = \frac{-(y_i - x_i)}{\sqrt{(at)^2 - (y_1-x_1)^2 - (y_2-x_2)^2}} \quad (i = 1.2)$$

从而 $$\sqrt{1 + |\nabla y_3|^2} = \frac{at}{\sqrt{(at)^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}$$

故 $$\iint_{S_{at}^+} \frac{\psi}{4\pi a^2 t} dS = \frac{1}{4\pi a} \iint_{S_{at}^+} \frac{\psi}{at} dS$$

$$= \frac{1}{4\pi a} \iint_{(y_1-x_1)^2 + (y_2-x_2)^2 \le (at)^2} \frac{\psi}{at} \sqrt{1 + |\nabla y_3|^2}\, dy_1\, dy_2$$

$$= \frac{1}{4\pi a} \iint_{(y_1-x_1)^2 + (y_2-x_2)^2 \le (at)^2} \frac{\psi(y_1 y_2)}{\sqrt{(at)^2 - (y_1-x_1)^2 - (y_2-x_2)^2}} dy_1\, dy_2$$

类似可计算 $\int_{S_{at}} \varphi(y)\, dS$ 的值

从而 $$u(x_1. x_2. t) = \frac{1}{2\pi a} \left(\iint_{\Sigma_{at}} \frac{\psi(y_1. y_2)}{\sqrt{(at)^2 - r^2}} d\, y_1 y_2 \right.$$

$$\left. + \frac{\partial}{\partial t} \iint_{\Sigma_{at}} \frac{\varphi(y_1. y_2)}{\sqrt{(at)^2 - r^2}} dy_1\, dy_2 \right)$$

其中 $r^2=(x_1-y_1)^2+(x_2-y_2)^2$

$\Sigma_{at}=\{y\in\mathbb{R}^2 \mid (y_1-x_1)^2+(y_2-x_2)^2\le(at)^2\}$.

上式也可写为 $\frac{1}{2\pi a}\left(\int_0^{at}\int_0^{2\pi}\frac{\psi(x_1+r\cos\theta,\ x_2+r\sin\theta)}{\sqrt{(at)^2-r^2}}r\,d\theta\,dr\right.$

$\left.+\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{\varphi(x_1+r\cos\theta,\ x_2+r\sin\theta)}{\sqrt{(at)^2-r^2}}r\,d\theta\,dr\right)$

以上公式称为 Poisson 公式.

注: Poisson 公式可解释水波的传播问题.

非齐次方程 (Duhamel 原理)

考虑方程 $\begin{cases} u_{tt}-a^2\Delta u=f(x,t) & x\in\mathbb{R}^3,\ t>0 \\ u|_{t=0}=0,\quad u_t|_{t=0}=0 \end{cases}$.

命题: 若 $w(x,t;\tau)$ 为问题

$$\begin{cases} w_{tt}-a^2\Delta_x w(x,t;\tau)=0 & x\in\mathbb{R}^3,\ t>\tau \\ w|_{t=\tau}=0,\quad w_t|_{t=\tau}=f(x,\tau) & x\in\mathbb{R}^3 \end{cases}.$$

的解, 则 $u(x,t)=\int_0^t w(x,t;\tau)\,d\tau$ 为原方程解.

证明: $u|_{t=0}=0$.

$$u_t=w(x,t;t)+\int_0^t\frac{\partial w}{\partial t}(x,t;\tau)\,d\tau$$

$$u_t|_{t=0}=w(x,0;0)=0$$.

且 $u_{tt}=\frac{\partial w}{\partial t}(x,t;\tau)\Big|_{\tau=t}+\int_0^t\frac{\partial^2 w}{\partial t^2}(x,t;\tau)\,d\tau$

$=f(x,t)+a^2\Delta_x\int_0^t w(x,t;\tau)\,d\tau$ 得证.

下面求解 w 的方程

令 $v(x,t;\tau)=w(x,t+\tau;\tau)$

则 v 满足 $\begin{cases} v_{tt}-a^2\Delta_x v(x,t;\tau)=0 \\ v(x,0;\tau)=0 \\ v_t(x,0;\tau)=f(x,\tau) \end{cases}$

由Kirchhoff 公式有

$$v(x,t;\tau)=\frac{1}{4\pi a^2 t}\int_{|y-x|=at} f(y,\tau)\,dS_y$$

代入有 $u(x,t)=\int_0^t v(x,t-\tau;\tau)\,d\tau$

$$=\frac{1}{4\pi a^2}\int_0^t \frac{d\tau}{t-\tau}\int_{|y-x|=a(t-\tau)} f(y,\tau)\,dS_y$$

$$=\frac{1}{4\pi a^2}\int_0^{at} dr\int_{|y-x|=r}\frac{f(y,t-\frac{r}{a})}{r}\,dS_y$$

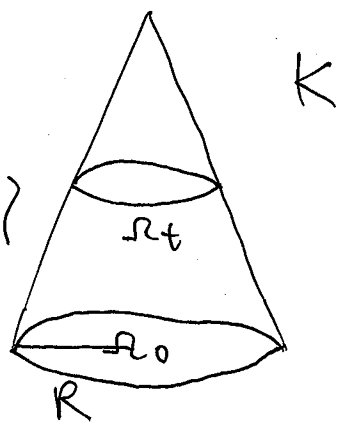
其中 $r=a(t-\tau)$. 原方程解为

$$u(x,t)=\frac{1}{4\pi a^2}\int_{|y-x|\le at}\frac{f(y,t-\frac{|y-x|}{a})}{|y-x|}\,dy$$

上述表达式中 $\frac{|y-x|}{a}$ 被称为推迟势

§12.4 能量积分

在如图所示的特征锥K中



$\Omega_t \triangleq \{(x,y)\mid (x-x_0)^2+(y-y_0)^2\le (R-at)^2\}$

$\Omega_0=\{(x,y)\mid (x-x_0)^2+(y-y_0)^2\le R^2\}$

$$E(t)\triangleq \frac{1}{2}\iint_{\Omega_t} u_t^2+a^2|\nabla u|^2\,dx\,dy$$

定理1. $u_{tt}-a^2(u_{xx}+u_{yy})=0$

对应的 $E(t)$ 满足 $E(t)\le E(0)$.

证明：只要证 $E(t)$ 在 $(0,\frac{R}{a})$ 递减.

$$\frac{dE(t)}{dt}=\frac{d}{dt}\iint_{(x-x_0)^2+(y-y_0)^2\le(R-at)^2}\frac{1}{2}\left[u_t^2+a^2(u_x^2+u_y^2)\right]dxdy$$

$$=\frac{d}{dt}\int_0^{R-at}\int_0^{2\pi r}\left[u_t^2+a^2(u_x^2+u_y^2)\right]ds\,dr.$$

其中 $r=\sqrt{(x-x_0)^2+(y-y_0)^2}$ $\quad ds=r\,d\theta$.

$$上式=\int_0^{R-at}\int_0^{2\pi r}\left[u_tu_{tt}+a^2(u_xu_{xt}+u_yu_{yt})\right]ds\,dr$$

$$-\frac{a}{2}\int_{\Gamma_t}\left[u_t^2+a^2(u_x^2+u_y^2)\right]ds\quad(其中\ \Gamma_t=\partial\Omega_t)$$

$$=\int_0^{R-at}\int_0^{2\pi r}\underset{①}{u_t\left[u_{tt}-a^2(u_{xx}+u_{yy})\right]}ds\,dr$$

$$+\int_{\Gamma_t}\Big\{\underset{②}{a^2\left(u_xu_t\cos\angle(\vec{n},\vec{x})+u_yu_t\cos\angle(\vec{n},\vec{y})\right)}$$

$$-\frac{a}{2}\left[u_t^2+a^2(u_x^2+u_y^2)\right]\Big\}ds$$

由 ①$=0$. ② 中被积函数

$$a^2\left[u_xu_t\cos\angle(\vec{n},\vec{x})+u_yu_t\cos\angle(\vec{n},\vec{y})\right]-\frac{a}{2}\left[u_t^2+a^2(u_x^2+u_y^2)\right]$$

$$=-\frac{a}{2}\left[\left(au_x-u_t\cos\angle(\vec{n},\vec{x})\right)^2+\left(au_y-u_t\cos\angle(\vec{n},\vec{y})\right)^2\right]\le0$$

从而 $\dfrac{dE}{dt}=①+②\le0$. 即 $E(t)\downarrow$.

故有 $E(t)\le E(0)$ $\quad(t>0)$

下面用能量法证明高维情形的唯一性定理

定理2：
$$\begin{cases} u_{tt}-a^2(u_{xx}+u_{yy})=f & x\in\Omega \quad t\in[0,T] \\ u|_{t=0}=\varphi \quad u_t|_{t=0}=\psi \\ u|_{\partial\Omega}=0 \end{cases}$$

的解若存在，则必然唯一．

证明：设 u_1, u_2 为两个解，$u\triangleq u_1-u_2$．

则
$$\begin{cases} u_{tt}=a^2(u_{xx}+u_{yy}) \quad \text{in } \Omega \\ u|_{t=0}=0 \quad u_t|_{t=0}=0 \\ u|_{\partial\Omega}=0 \end{cases}$$

设
$$E(t)\triangleq\iint_\Omega u_t^2+a^2(u_x^2+u_y^2)\,dx\,dy$$

则
$$E(0)=\iint_\Omega u_t^2+a^2(u_x^2+u_y^2)\Big|_{t=0}dx\,dy=0$$

$$\frac{dE}{dt}=2\iint_\Omega u_tu_{tt}+a^2(u_xu_{xt}+u_yu_{yt})\,dx\,dy$$

$$=2\iint_\Omega u_t(u_{tt}-a^2u_{xx}-u_{yy})\,dx\,dy$$

$$+2a^2\iint_\Omega (u_xu_t)_x+(u_yu_t)_y\,dx\,dy$$

$$=2a^2\iint_\Omega \mathrm{div}(u_t\cdot\nabla u)\,dx\,dy=2a^2\int_{\partial\Omega}u_t\frac{\partial u}{\partial n}\,ds$$

而 $u|_{\partial\Omega}=0\Rightarrow u_t|_{\partial\Omega}=0$ 故 $\dfrac{dE}{dt}\equiv0$．

$E(t)=E(0)=0$．从而 $u\equiv0$ in Ω

即 $u_1\equiv u_2$．这就证明了解的唯一性．

下面讨论波方程解的稳定性．

先证明两个能量不等式．

由 $\frac{dE}{dt}=2\iint_\Omega u_t u_{tt}+a^2(u_x u_{xt}+u_y u_{yt})\,dxdy$

$=2\iint_\Omega u_t[u_{tt}-a^2(u_{xx}+u_{yy})]\,dxdy$

$=2\iint_\Omega u_t f\,dxdy \leq \iint_\Omega u_t^2\,dxdy+\iint_\Omega f^2\,dxdy$

$\leq E(t)+\iint_\Omega f^2\,dxdy$

故 $e^{-t}\frac{dE}{dt}-e^{-t}E\leq e^{-t}\iint_\Omega f^2\,dxdy$

$\Rightarrow \frac{d}{dt}(e^{-t}E(t))\leq \int_0^t e^{-s}\iint_\Omega f^2(x,y,s)\,dxdy\,ds$

$\Rightarrow E(t)\leq e^t\left(E(0)+\int_0^t e^{-s}\iint_\Omega f^2\,dxdy\,ds\right)$

$\leq e^T\left(E(0)+\int_0^T\iint_\Omega f^2\,dxdy\,ds\right)$

再设 $E_0(t)\triangleq\iint_\Omega u^2\,dxdy$.

则 $\frac{dE_0}{dt}=2\iint_\Omega u u_t\,dxdy$

$\leq\iint_\Omega u^2\,dxdy+\iint_\Omega u_t^2\,dxdy\leq E_0(t)+E(t)$

故 $\frac{dE_0}{dt}-E_0\leq E \Rightarrow \frac{d}{dt}(e^{-t}E_0)\leq e^{-t}E(t)$

$\Rightarrow e^{-t}E_0(t)-E_0(0)\leq\int_0^t e^{-s}E(s)\,ds$

$\Rightarrow E_0(t)\leq e^tE_0(0)+e^t\int_0^t e^{-s}E(s)\,ds$

而 $\int_0^t e^{-s}E(s)\,ds\leq\int_0^t\left[E(0)+\int_0^T\iint_\Omega f^2\,dxdy\,ds\right]d\tau$

$=tE(0)+t\int_0^T\iint_\Omega f^2\,dxdy\,ds$

$\le T E(0) + T\int_0^T\iint_\Omega f^2\,dxdy\,ds$

从而 $E_0(t) \le e^T E_0(0) + Te^T\left(E(0)+\int_0^T\iint_\Omega f^2\,dxdy\,ds\right)$

再结合 $E(t) \le e^T\left(E(0)+\int_0^T\iint_\Omega f^2\,dxdy\,ds\right)$

知 $E_0(t)+E(t) \le C\left(E(0)+E_0(0)+\int_0^T\iint_\Omega f^2\,dxdy\,ds\right)$

其中 C 只依赖于 T. 例如可取 $C = Te^T + e^T$

定理：原方程的解在下述意义下稳定：

$\forall \varepsilon > 0$. $\exists \eta = \eta(\varepsilon, T)$ 只要 $\|\varphi_1-\varphi_2\|_{L^2} < \eta$

$\|\nabla\varphi_1-\nabla\varphi_2\|_{L^2(\Omega)} < \eta$ $\quad \|\psi_1-\psi_2\|_{L^2(\Omega)} < \eta$. $\|f_1-f_2\|_{L^2_{\Omega,t}} < \eta$

则 $\|u_1-u_2\|_{L^2(\Omega)} < \varepsilon$. $\|u_{1t}-u_{2t}\|_{L^2(\Omega)} < \varepsilon$.

且有 $\|\nabla u_1-\nabla u_2\|_{L^2(\Omega)} < \varepsilon$

其中 u_i 为方程 $\begin{cases} u_{tt} - a^2(u_{xx}+u_{yy}) = f_i \\ u|_{t=0} = \varphi_i \quad u_t|_{t=0} = \psi_i \\ u|_{\partial\Omega} = 0 \end{cases}$ 的解 $(i=1,2)$

证明：记 $v \triangleq u_1-u_2$. 则有

$$\begin{cases} v_{tt} = a^2(v_{xx}+v_{yy}) + f_1 - f_2 \\ v|_{t=0} = \varphi_1-\varphi_2 \quad v_t|_{t=0} = \psi_1-\psi_2 \\ v|_{\partial\Omega} = 0 \end{cases}$$

定义 $E(t) = \iint_\Omega v_t^2 + a^2(v_x^2+v_y^2)\,dxdy$

$$E_0(t) = \iint_\Omega v^2\,dxdy$$

则有 $E(0) = \iint_\Omega |\psi_1-\psi_2|^2 + a^2|\nabla\varphi_1-\nabla\varphi_2|^2\,dxdy$

$= \|\psi_1-\psi_2\|_{L^2}^2 + a^2\|\nabla\varphi_1-\nabla\varphi_2\|_{L^2}^2 < (a^2+1)\eta^2$

$$E_0(0)=\iint_\Omega |\varphi_1-\varphi_2|^2\,dx\,dy<\eta^2.$$

$$\int_0^T\iint_\Omega f^2\,dx\,dy\,ds=\int_0^T\iint_\Omega |f_1-f_2|^2\,dx\,dy\,ds$$

$$=\|f_1-f_2\|^2_{L^2_{\Omega,t}}<\eta^2.$$

由 $E(t)+E_0(t)\le C\left(E(0)+E_0(0)+\int_0^T\iint_\Omega f^2\,dx\,dy\,ds\right)$

$\le C\left((a^2+1)\eta^2+\eta^2+\eta^2\right)$ 知 η 充分小时有

$E(t)+E_0(t)<\varepsilon^2$. 由定义知有下式成立

$\|u_1-u_2\|_{L^2(\Omega)}<\varepsilon \quad \|\nabla u_1-\nabla u_2\|_{L^2(\Omega)}<\varepsilon.\quad \|u_{1t}-u_{2t}\|_{L^2(\Omega)}<\varepsilon$

注：对函数 u. $\|u\|_{L^2(\Omega)}\triangleq\sqrt{\iint_\Omega u^2\,dx\,dy}$

在上面的证明中. 将 $\|f_1-f_2\|_{L^2_{\Omega,t}}$ 定义为

$$\left(\int_0^T\iint_\Omega |f_1-f_2|^2\,dx\,dy\,ds\right)^{\frac{1}{2}}$$

$\|\cdot\|_{L^2}$ 称为函数的 L^2 范数.

$L^2(\Omega)$ 表示 Ω 上一切平方可积函数构成的函数空间.

在实变函数与泛函分析的课程中将会说明

$L^2(\Omega)$ 是一类典型的 Hilbert 空间.

而可积的定义也会推广到勒贝格积分的情形.

一般的. 定义 $\|f\|_{L^p(\Omega)}=\left(\int_\Omega |f|^p\,dx\right)^{\frac{1}{p}}\quad (p>1)$

称为 f 的 L^p 范数.

第十三章 热传导方程

§13.1 Fourier 变换及其性质

定义：设 $f(x)$ 在 $\mathbb{R}^n$ 上连续可微且绝对可积

则有 $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$

称 $\hat{f}(\xi)$ 为 $f(x)$ 的 Fourier 变换

$\hat{f}(\xi)$ 的逆变换为 $f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\xi} d\xi$

其中，内积 $x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n$

定理：Fourier 变换满足下列性质

(1) $\widehat{f'_{x_j}} = i\xi_j \hat{f}$

(2) $(-ix_j f)^\wedge = \frac{\partial}{\partial \xi_j} \hat{f}$

(3) 定义卷积 $(f*g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y) dy$

则 $\widehat{f*g} = \hat{f}\cdot\hat{g}$

(4) $(fg)^\vee = \check{f} * \check{g}$.

证明：(1) 只要证一元函数情形即可

$\hat{f'}(\xi) = \int_{\mathbb{R}} f'(x) e^{-ix\xi} dx = -\int_R f(x)(-i\xi) e^{-ix\xi} dx$

$= i\xi \int_R f(x) e^{-ix\xi} dx = i\xi \hat{f}(\xi)$

(2) 由 $\widehat{-ixf}(\xi) = \int_R (-ixf(x)) e^{-ix\xi} dx$

又 $\frac{\partial}{\partial\xi}\hat{f}(\xi) = \frac{\partial}{\partial\xi}\int_R f(x) e^{-ix\xi} dx$

$= -i\int_R x f(x) e^{-ix\xi} dx =$ 左边　　　得证

(3) $\widehat{f*g}(\xi) = \int_R (f*g)(x)\, e^{-ix\xi}\, dx$

$= \int_R \int_R f(y) g(x-y)\, e^{-ix\xi}\, dy\, dx$

$= \int_R \int_R f(y) g(x-y)\, e^{-iy\xi - i(x-y)\xi}\, dx\, dy$

$= \int_R \left(\int_R g(x-y) e^{-i(x-y)\xi} dx \right) f(y) e^{-iy\xi} dy$

$= \int_R \hat{g}(\xi) f(y) e^{-iy\xi} dy = \hat{f}(\xi)\hat{g}(\xi)$

(4) 由 (3) 知 $\widehat{f_1 * g_1} = \hat{f_1} \cdot \hat{g_1}$. 作逆变换得

$f_1 * g_1 = (\hat{f_1} \cdot \hat{g_1})^{\vee}$ 令 $f = \hat{f_1}$, $g = \hat{g_1}$

则有 $\check{f} * \check{g} = (fg)^{\vee}$ 得证

例 1: 求 $e^{-a|x|}$ 的 Fourier 变换

解: $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-a|x|} e^{-ix\xi} dx$

$= \int_{\mathbb{R}} e^{-a|x|} (\cos x\xi - i \sin x\xi) dx = \int_{\mathbb{R}} e^{-a|x|} \cos x\xi\, dx$

$= 2\int_0^{+\infty} e^{-ax} \cos x\xi\, dx$

记 $I = \int_0^{+\infty} e^{-ax} \cos x\xi\, dx = \frac{\sin x\xi}{\xi} e^{-ax}\Big|_0^{+\infty}$

$+ \int_0^{+\infty} \frac{\sin x\xi}{\xi} a e^{-ax} dx = \frac{a}{\xi} \int_0^{+\infty} \sin x\xi\, e^{-ax} dx$

$= \frac{a}{\xi} \left(-\frac{\cos x\xi}{\xi}\Big|_0^{+\infty} + \int_0^{+\infty} \frac{\cos x\xi}{\xi} (-a) e^{-ax} dx \right)$

$= -\frac{a^2}{\xi^2} I + \frac{a}{\xi^2} \Rightarrow I = \frac{a}{a^2+\xi^2} \Rightarrow \hat{f}(\xi) = \frac{2a}{a^2+\xi^2}$

例2：求 $f(\xi)=e^{-|\xi|^2 t}$ 的Fourier逆变换

解：$\check{f}(x)=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n} e^{-|\xi|^2 t+ix\xi}\,d\xi$

$$=\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t\xi_k^2+ix_k\xi_k}\,d\xi_k\right)^n$$

$$=\left(\frac{1}{\pi}\int_0^{+\infty} e^{-t\xi_k^2}\cos x_k\xi_k\,d\xi_k\right)^n$$

记 $I(x_k)=\frac{1}{\pi}\int_0^{+\infty} e^{-t\xi_k^2}\cos x_k\xi_k\,d\xi_k$

则 $-I'(x_k)=\frac{1}{\pi}\int_0^{+\infty} e^{-t\xi_k^2}\xi_k\sin(x_k\xi_k)\,d\xi_k$

$$=\frac{1}{\pi}\left[-\frac{1}{2t}e^{-t\xi_k^2}\sin(x_k\xi_k)\Big|_0^{+\infty}+\int_0^{+\infty}\frac{1}{2t}e^{-t\xi_k^2}\cos(x_k\xi_k)\,x_k\,d\xi_k\right]$$

$$=\frac{x_k}{2t\pi}\int_0^{+\infty} e^{-t\xi_k^2}\cos(x_k\xi_k)\,d\xi_k=\frac{x_k}{2t}I(x_k)$$

$$\Rightarrow \frac{dI}{dx_k}+\frac{x_k}{2t}I=0 \Rightarrow I=ce^{-\frac{x_k^2}{4t}}$$

由 $I(0)=\frac{1}{\pi}\int_0^{+\infty} e^{-t\xi_k^2}\,d\xi_k=\frac{1}{2}\sqrt{\frac{1}{\pi t}}$ 知

$$I(x_k)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x_k^2}{4t}} \Rightarrow \check{f}(x)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

§13.2 初值问题的解

考虑方程 $\begin{cases} u_t-\Delta u=0 & t>0 \\ u|_{t=0}=\varphi \end{cases}$

作 Fourier 变换得 $\begin{cases} \hat{u}_t+|\xi|^2\hat{u}=0 \\ \hat{u}|_{t=0}=\hat{\varphi} \end{cases}$

$$\Rightarrow \hat{u}=\hat{u}|_{t=0}\cdot e^{-|\xi|^2 t}=\hat{\varphi}(\xi)e^{-|\xi|^2 t}$$

从而 $u(x,t)=(\hat{u}(\xi,t))^{\vee}=\left(\hat{\varphi}(\xi)e^{-|\xi|^2 t}\right)^{\vee}$

$$= \varphi * \left(e^{-|\xi|^2 t}\right)^{\vee} = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(y) e^{-\frac{|x-y|^2}{4t}} dy$$

记此式为 $\int_{\mathbb{R}^n} K(x-y, t) \varphi(y) dy$

（以上利用了上节例 2 的结论）.

其中 $K(x-y, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$ 称为热核.

故 $u(x, t) = \int_{\mathbb{R}^n} \varphi(y) K(x-y, t) dy$ 为解的表达式.

下面列举出热核的几条性质：

(1) $K(x-y, t) > 0$，$K(x-y, t) \in C^{\infty}$

(2) $\left(\frac{\partial}{\partial t} - \Delta\right) K(x-y, t) = 0$.

(3) $\int_{\mathbb{R}^n} K(x-y, t) dy = 1 \quad \forall x \in \mathbb{R}^n, \ t > 0$

(4) $\forall \delta > 0$，有 $\lim\limits_{t \to 0^+} \int_{|x-y| > \delta} K(x-y, t) dy = 0$

证明：(1) 显然

(2) 只要证 $\left(\frac{\partial}{\partial t} - \Delta\right)\left(t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}\right) = 0$

$$\frac{\partial}{\partial t}\left(t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}\right) = -\frac{n}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} + \frac{1}{4} t^{-\frac{n}{2}-2} e^{-\frac{|x|^2}{4t}} |x|^2$$

$$\Delta\left(t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}\right) = t^{-\frac{n}{2}} \left(\frac{-x_i}{2t} e^{-\frac{|x|^2}{4t}}\right)_{x_i}$$

$$= t^{-\frac{n}{2}} \left(-\frac{n}{2t} e^{-\frac{|x|^2}{4t}}\right) - \frac{x_i}{2} t^{-\frac{n}{2}-1} \left(-\frac{x_i}{2t}\right) e^{-\frac{|x|^2}{4t}}$$

$$= -\frac{n}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} + \frac{1}{4} t^{-\frac{n}{2}-2} e^{-\frac{|x|^2}{4t}} \cdot |x|^2$$

从而知 $\left(\frac{\partial}{\partial t} - \Delta\right)\left(t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}\right) = 0$

即有 $\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}\right) = 0$

(3) 令 $y = x + (4t)^{\frac{1}{2}} \eta$

则有 $\int_{\mathbb{R}^n} K(x-y,t)\,dy = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|\eta|^2}\,dy$

$= \frac{1}{\pi^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|\eta|^2}\,d\eta = 1$

(4) 同(3)令 $y = x + (4t)^{\frac{1}{2}}\eta$，则有

$\int_{|y-x|>\delta} K(x-y,t)\,dy = \int_{|\eta|>\frac{\delta}{\sqrt{4t}}} e^{-|\eta|^2}\,d\eta$

$= C_n \int_{\frac{\delta}{\sqrt{4t}}}^{+\infty} e^{-r^2} r^{n-1}\,dr$

$\because \int_0^{+\infty} e^{-r^2} r^{n-1}\,dr < +\infty$，且 $\lim\limits_{t\to 0^+} \frac{\delta}{\sqrt{4t}} \to +\infty$.

$\therefore \lim\limits_{t\to 0^+} \int_{|y-x|>\delta} K(x-y,t)\,dy = 0$.

注：一般的 Riemann 流形上也可以定义热核的概念.
详细的讨论参考《微分几何讲义》. 丘成桐, 孙理察.

下面我们证明 $u(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}}\int \varphi(y)\, e^{-\frac{|x-y|^2}{4t}}\,dy$

的确给出了一维热方程 $u_t = u_{xx}$ 的解.（$|\varphi| \le M$ 情形）

Step 1：$|u(x,t)| = \frac{1}{(4\pi t)^{\frac{1}{2}}}\left|\int_{\mathbb{R}^1} \varphi(y)\, e^{-\frac{|x-y|^2}{4t}}\,dy\right|$

$\le \frac{M}{(4\pi t)^{\frac{1}{2}}}\int_{\mathbb{R}^1} e^{-\frac{|x-y|^2}{4t}}\,dy = \frac{M}{(4\pi t)^{\frac{1}{2}}}\int_{\mathbb{R}^1} e^{-\eta^2}\,d(\sqrt{4t}\,\eta)$

$= \frac{M}{\pi^{\frac{1}{2}}}\int_{\mathbb{R}^1} e^{-\eta^2}\,d\eta = M$ 从而收敛

Step 2：考虑 $\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} \frac{-(x-y)\varphi(y)}{2t^{\frac{3}{2}}}\, e^{-\frac{(x-y)^2}{4t}}\,dy$

易见其在 $t \ge t_0 > 0$ 范围内是一致收敛的.

因此 $t>0$ 时成立

$$\frac{\partial u}{\partial x}(x,t)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{-(x-y)\varphi(y)}{2t^{\frac{3}{2}}}e^{-\frac{|x-y|^2}{4t}}dy$$

同理可证 $\frac{\partial}{\partial t}$ 也与积分号可交换

Step 3：验证 $u_t=u_{xx}$

由 $$u_t=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(y)e^{-\frac{|x-y|^2}{4t}}\cdot\frac{|x-y|^2}{4t^2}dy$$

$$-\frac{1}{4}\frac{1}{\sqrt{\pi}}t^{-\frac{3}{2}}\int_{-\infty}^{+\infty}\varphi(y)e^{-\frac{|x-y|^2}{4t}}dy$$

$$u_{xx}=\frac{-1}{4\sqrt{\pi}}t^{-\frac{3}{2}}\int_{-\infty}^{+\infty}\left((x-y)e^{-\frac{|x-y|^2}{4t}}\right)_x\varphi(y)dy$$

$$=-\frac{1}{4\sqrt{\pi}}t^{-\frac{3}{2}}\int_{-\infty}^{+\infty}\left[e^{-\frac{|x-y|^2}{4t}}\varphi(y)-\frac{(x-y)^2}{2t}e^{-\frac{|x-y|^2}{4t}}\varphi(y)\right]dy$$

$$=-\frac{1}{4\sqrt{\pi}}t^{-\frac{3}{2}}\int_{-\infty}^{+\infty}e^{-\frac{|x-y|^2}{4t}}\varphi(y)dy$$

$$+\frac{1}{8\sqrt{\pi}}t^{-\frac{5}{2}}\int_{-\infty}^{+\infty}|x-y|^2e^{-\frac{|x-y|^2}{4t}}\varphi(y)dy$$

故有 $u_t=u_{xx}$ 成立.

Step 4：验证 $\lim\limits_{\substack{x\to x_0\\ t\to 0}}u(x,t)=\varphi(x_0)$

即证 $\forall\varepsilon>0$，$\exists\delta>0$，$|x-x_0|<\delta$ 且 $t<\delta$ 时

有 $|u(x,t)-\varphi(x_0)|<\varepsilon$

$$u(x,t)=\frac{1}{(4\pi t)^{\frac{1}{2}}}\int_{\mathbb{R}^1}\varphi(y)e^{-\frac{|x-y|^2}{4t}}dy$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\varphi(x+2a\sqrt{t}\xi)e^{-\xi^2}d\xi$$

又有 $$\varphi(x_0)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\varphi(x_0)e^{-\xi^2}d\xi$$

从而 $u(x,t)-\varphi(x_0)$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}[\varphi(x+2a\sqrt{t}\xi)-\varphi(x_0)]e^{-\xi^2}d\xi$$

取 N 充分大. s.t. $\frac{1}{\sqrt{\pi}}\int_N^{\infty}e^{-\xi^2}d\xi<\frac{\varepsilon}{6M}$.

$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-N}e^{-\xi^2}d\xi<\frac{\varepsilon}{6M}$. 固定 N. $\exists\delta>0$.

当 $|x-x_0|<\delta$. $t<\delta$ 时有

$|\varphi(x+2a\sqrt{t}\xi)-\varphi(x_0)|<\frac{\varepsilon}{3}\quad(-N<\xi<N)$

因此 $|u(x,t)-\varphi(x_0)|=\left|\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}[\varphi(x+2a\sqrt{t}\xi)-\varphi(x_0)]e^{-\xi^2}d\xi\right|$

$$\le\frac{1}{\sqrt{\pi}}\int_{-N}^{N}|\varphi(x+2a\sqrt{t}\xi)-\varphi(x_0)|e^{-\xi^2}d\xi$$

$$+\frac{2M}{\sqrt{\pi}}\int_{-\infty}^{-N}e^{-\xi^2}d\xi+\frac{2M}{\sqrt{\pi}}\int_N^{\infty}e^{-\xi^2}d\xi$$

$$\le\frac{\varepsilon}{3}\frac{1}{\sqrt{\pi}}\int_{-N}^{N}e^{-\xi^2}d\xi+4M\cdot\frac{\varepsilon}{6M}<\varepsilon.$$

从而我们验证了上述表达式确实是 $u_t=u_{xx}$ 的解

一般的. 可以验证 $u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\varphi(y)e^{-\frac{|x-y|^2}{4t}}dy$

为 n 维热方程 $u_t=\Delta u$ 的解 (留作习题)

§13.3 最大值原理

定理1: 设 $u(x,t)\in C^{2,1}(Q_T)\cap C(\overline{Q_T})$. 且

为 $u_t-\Delta u=0$ 在 Q_T 上的下解. 则有

$$\max_{\overline{Q_T}}u(x,t)=\max_{\Gamma_T}u(x,t).$$

其中 Γ_T 为 Q_T 的抛物边界

证明: 先设 $u_t-\Delta u<0$. 因 $u\in C(\overline{Q_T})$

故 $\exists(x_0,t_0)\in\overline{Q_T}$ s.t. $u(x_0,t_0)=\max_{\overline{Q_T}}u$

若 $(x_0, t_0) \in Q_T$ 则 $u_t(x_0, t_0) = 0$

且 $\Delta u(x_0, t_0) \le 0$. 故 $u_t - \Delta u \big|_{(x_0, t_0)} \ge 0$. 矛盾.

从而 $\max\limits_{\overline{Q_T}} u = \max\limits_{\Gamma_T} u$

Step 2 (证法一). 设 $v = u - \varepsilon t$. 由 $u_t - \Delta u \le 0$.

则 $v_t = u_t - \varepsilon$, $\Delta v = \Delta u$. $v_t - \Delta v \le -\varepsilon < 0$.

从而由 Step 1 有 $\max\limits_{\overline{Q_T}} v = \max\limits_{\Gamma_T} v$

$$\Rightarrow \max_{\overline{Q_T}} u \le \max_{\overline{Q_T}} v + \varepsilon T \le \max_{\Gamma_T} v + \varepsilon T$$

$\le \max\limits_{\Gamma_T} u + \varepsilon T$ 令 $\varepsilon \to 0^+$ 有 $\max\limits_{\overline{Q_T}} u \le \max\limits_{\Gamma_T} u$.

(证法二). 设 $M \triangleq \max\limits_{\overline{Q_T}} u$. $m = \max\limits_{\Gamma_T} u$ 且 $M > m$

则 $\exists x^* \in \Omega$ $0 < t^* < T$ $u(x^*, t^*) = M$

令 $v(x, t) = u(x, t) + \dfrac{M - m}{4d^2} |x - x^*|^2$ $(d = \mathrm{diam}(\Omega))$

易见 $\Delta v = \Delta u + \dfrac{M - m}{4d^2} \cdot 2n > \Delta u$.

从而 $v_t - \Delta v < u_t - \Delta u \le 0$. 故由 Step 1 知

$v(x, t)$ 不在内部取极大值. 而另一方面在 Γ_T 上有

$v(x, t) < m + \dfrac{M - m}{4d^2} \cdot d^2 = \dfrac{M}{4} + \dfrac{3m}{4} < M$.

且 $v(x^*, t^*) = u(x^*, t^*) = M$. 矛盾. 故 $M = m$.

即 $\max\limits_{\overline{Q_T}} u = \max\limits_{\Gamma_T} u$. 证毕.

以上用两种方法证明了热方程的弱极大值原理

关于强极值原理的讨论参考《偏微分方程》陈祖墀

例1：设 u 满足 $\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega\times(0,T] \\ u = \varphi(t) & \text{on } \partial\Omega \\ u|_{t=0} = g(x) \end{cases}$

定义 $G(x,t) \triangleq 2tu_t + \sum_{i=1}^{n} x_i u_i$ 求证 $\Delta G - G_t = 0$.

证：$G_t = 2u_t + 2tu_{tt} + \sum_{i=1}^{n} x_i u_{it}$

$\Delta G = 2t\Delta u_t + \sum_{i=1}^{n} (x_i u_i)_{ll}$

$= 2t\Delta u_t + 2\delta_{il} u_{il} + x_i u_{ill}$

$= 2t\Delta u_t + 2\Delta u + \sum_i x_i \Delta u_i$

$= 2t u_{tt} + 2u_t + \sum_{i=1}^{n} x_i u_{ti}$

从而有 $\Delta G = G_t$

利用弱极值原理可直接得到解的唯一性定理（证明从略）

下面我们讨论如下方程解的稳定性

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \Omega\times(0,T] \\ u = g(t) & \text{on } \partial\Omega \\ u|_{t=0} = \varphi(x) \end{cases}$$

定理2：设 $F \triangleq \max_{\overline{Q}_T} |f|$ $\quad B \triangleq \max\{\Phi, G\}$.

其中 $\Phi = \sup_x |\varphi(x)|$ ，$G = \sup_t |g(t)|$

则 $\max_{\overline{Q}_T} |u| \le FT + B$.

证明：设 $w(x,t) \triangleq u(x,t) - Ft - B$.

则 $w_t = u_t - F$. $\Delta w = \Delta u$. 故有

$w_t - \Delta w = u_t - \Delta u - F = f - F \le 0$.

又 $w|_{\Gamma_T} = u|_{\Gamma_T} - Ft - B \le u|_{\Gamma_T} - B \le 0$.

从而 $w \le 0$ in $\overline{Q_T}$. 即 $u \le FT + B$.

若令 $v \triangleq -u + Ft + B$ 类似可得 $-u \ge -FT - B$.

故 $\max\limits_{\overline{Q_T}} |u| \le FT + B$. 定理得证.

而另一方面. 若 u_1, u_2 分别为方程

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \Delta u_i = f_i & \text{in } \overline{Q_T} \\ u_i = g_i(t) & \text{on } \partial\Omega \\ u_i|_{t=0} = \varphi_i(x) \end{cases} \quad (i = 1, 2) \text{ 的解}$$

则若令 $v \triangleq u_1 - u_2$. 则有 $\begin{cases} v_t - \Delta v = f_1 - f_2 & \text{in } \overline{Q_T} \\ v = g_1 - g_2 & \text{on } \partial\Omega \\ v|_{t=0} = \varphi_1 - \varphi_2 \end{cases}$

结合定理 2 知 $\max\limits_{\overline{Q_T}} |v| \le FT + B$.

即 $\max\limits_{\overline{Q_T}} |u_1 - u_2| \le T \max|f_1 - f_2| + \max\{G, \Phi\}$

$= T\|f_1 - f_2\|_{C^0} + \max\{\|g_1 - g_2\|_{C^0} + \|\varphi_1 - \varphi_2\|_{C^0}\}$

由此即得 热方程 的稳定性.

下面我们讨论全空间 $\mathbb{R}^n$ 上热方程解的唯一性.

定理 3: $\begin{cases} u_t - u_{xx} = f(x, t) \\ u|_{t=0} = \varphi(x) \quad (-\infty < x < \infty) \end{cases}$

的有界解 (即 $|u(x,t)| \le M$) 是唯一的.

证明: 只要证 $\begin{cases} u_t - u_{xx} = 0 \\ u|_{t=0} = 0 \quad (-\infty < x < \infty) \end{cases}$ 只有零解

$\forall (x_0, t_0)$ 满足 $t_0 > 0$. 考虑矩形

$R_0 : 0 \le t \le t_0 \quad |x - x_0| \le L$

$$v(x,t) \triangleq \frac{4M}{L^2}\left(\frac{(x-x_0)^2}{2} + t\right)$$

易见 $v_t = \frac{4M}{L^2}$ $\Delta v = \frac{4M}{L^2}$ 故 $v_t - \Delta v = 0$.

又 $v|_{t=0} = \frac{2M}{L^2}(x-x_0)^2 \geq 0 = u(x,0)$.

$v(x_0 \pm L, t) \geq 2M \geq u(x_0 \pm L, t)$

由极值原理知 $v(x,t) \geq u(x,t)$

即 $\frac{4M}{L^2}\left(\frac{(x-x_0)^2}{2} + t\right) \geq u$

同理可证 $u \geq -\frac{4M}{L^2}\left(\frac{(x-x_0)^2}{2} + t\right)$

故有 $|u(x_0,t_0)| \leq \frac{4M}{L^2} t_0$. 令 $L \to +\infty$ 得

$u(x_0, t_0) = 0$. 由 (x_0, t_0) 任意性知 $u \equiv 0$.

§13.4 能量积分

考虑方程 $\begin{cases} u_t = \Delta u + f(x,t) \\ u|_{\partial\Omega} = 0 \\ u|_{t=0} = \varphi(x) \end{cases}$ 定义 $E(t) = \int_\Omega u^2 dx$

方程两边同乘 u 有 $uu_t = u\Delta u + fu$.

$$\Rightarrow \frac{dE}{dt} = 2\int_\Omega uu_t dx = 2\int_\Omega u\Delta u dx + 2\int_\Omega fu$$

$$\leq -2\int_\Omega |\nabla u|^2 dx + \int_\Omega f^2 dx + \int_\Omega u^2 dx \quad \cdots (*)$$

$$\Rightarrow \frac{dE}{dt} \leq E(t) + \int_\Omega f^2 dx$$ 令 $\tilde{E}(t) = e^{-t}E(t)$.

则有 $$\frac{d\tilde{E}}{dt} \leq e^{-t}\int_\Omega f^2 dx \leq \int_\Omega f^2 dx$$

$$\Rightarrow \tilde{E}(t) - \tilde{E}(0) \leq \int_0^t \int_\Omega f^2 dx dt$$

$$\Rightarrow \tilde{E}(t) \leq \int_0^t \int_\Omega f^2 dx dt + \int_\Omega \varphi^2 dx$$

$$\Rightarrow E(t) \leq e^t\left(\int_0^T \int_\Omega f^2 dx dt + \int_\Omega \varphi^2 dx\right) \cdots (**)$$

将(*)式两边从0到T积分得

$$E(T)-E(0)+\int_0^T\int_\Omega |\nabla u|^2\,dx\,dt \le \int_0^T\int_\Omega (f^2+u^2)\,dx\,dt$$

$$\Rightarrow \int_0^T\int_\Omega |\nabla u|^2\,dx\,dt \le \int_0^T\int_\Omega (u^2+f^2)\,dx\,dt+\int_\Omega \varphi^2\,dx$$

$$=\int_0^T E(t)\,dt+\int_0^T\int_\Omega f^2\,dx\,dt+\int_\Omega \varphi^2\,dx$$

$$\overset{(**)}{\le}\int_0^T e^t\,dt\left(\int_0^T\int_\Omega f^2\,dx\,dt+\int_\Omega \varphi^2\,dx\right)$$

$$+\int_0^T\int_\Omega f^2\,dx\,dt+\int_\Omega \varphi^2\,dx$$

$$= e^T\left(\int_0^T\int_\Omega f^2\,dx\,dt+\int_\Omega \varphi^2\,dx\right)$$

推论：原方程的解若存在则唯一．

事实上，设 u_1, u_2 为两个解，$u \triangleq u_1-u_2$，则

$u|_{\partial\Omega}=0$，$u_t-\Delta u=0$，从而由此知

$$\int_0^T\int_\Omega |\nabla u|^2\,dx\,dt \le 0 \Rightarrow |\nabla u|\equiv 0 \quad \text{in } \Omega\times[0,T].$$

于是 $u\equiv \text{const}$，又 $u|_{\partial\Omega}=0$，故 $u\equiv 0$

即 $u_1\equiv u_2$．这就证明了解的唯一性

利用能量不等式还可以讨论解的稳定性

易见 $\|u_1-u_2\|_{L^2_{t,x}} \le C(T)\left(\|f\|_{L^2_{t,x}}+\|\varphi\|_{L^2_x}\right)$

第十四章 椭圆方程

§14.1 Green 函数及其应用

由散度定理得 $\int_\Omega \operatorname{div}(u\nabla v)\,dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\,dS$

$$\Rightarrow \int_\Omega (\nabla u\nabla v + u\Delta v)\,dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\,dS$$

同理有 $\int_\Omega (\nabla v\nabla u + v\Delta u)\,dx = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,dS$

故 $\int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$

上式称为第二格林公式. 下面设 $\Omega\subseteq\mathbb{R}^3$. 设 $y_0\in\Omega$

记 $\Omega_\varepsilon = \Omega\setminus B_\varepsilon(x_0)$ （ε充分小） 定义 $v\triangleq \frac{1}{4\pi}\frac{1}{|x-y_0|}$

则 $\int_{\Omega_\varepsilon}(u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$

$$= \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS - \int_{\partial B_\varepsilon(y_0)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS \cdots (*)$$

而 $\left|\int_{\partial B_\varepsilon(y_0)} v\frac{\partial u}{\partial n}\,dS\right| = \left|\int_{\partial B_\varepsilon(y_0)} \frac{1}{4\pi\varepsilon}\frac{\partial u}{\partial n}\,dS\right|$

$$\le \frac{1}{4\pi\varepsilon}\int_{\partial B_\varepsilon(y_0)} C_1\,dS = C\varepsilon \to 0 \quad (\varepsilon\to 0^+)$$

其中 $C_1 \triangleq \max\limits_{\partial\Omega}|Du| < +\infty$

又 $\int_{\partial B_\varepsilon(y_0)}(-u)\frac{\partial v}{\partial n} = \int_{\partial B_\varepsilon(y_0)} u\frac{\partial v}{\partial \rho}\,dS$

$$= \frac{1}{4\pi}\int_{\rho=\varepsilon} u\cdot\left(-\frac{1}{\rho^2}\right)dS = -\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon(y_0)} u\,dS$$

$\varepsilon\to 0^+$ 时 上式 $\to -u(y_0)$. 从而在 $(*)$ 式中令 $\varepsilon\to 0^+$

再结合 $\Delta v = 0$ in Ω_ε 可得

$$\int_\Omega -v\Delta u\,dx = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS - u(y_0)$$

$$\Rightarrow u(y_0) = \int_\Omega v\Delta u\,dx + \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$$

结合 y_0 的任意性我们得到如下定理.

定理1：设 $u\in C^2(\Omega)\cap C(\bar{\Omega})$，$\Omega$ 为有界区域.

设 $v \triangleq \frac{1}{4\pi}\frac{1}{|x-y|}$ 则可以得到 u 的表达式

$$u(y) = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS + \int_\Omega v(x-y)\Delta u\,dx$$

由上述表达式知：$u\big|_{\partial\Omega}$，$\frac{\partial u}{\partial n}\big|_{\partial\Omega}$，$\Delta u$ 的值可以完全确定 u 的值.

下面设法消去 $\frac{\partial u}{\partial n}$. 取函数 $h(x,y)$ 使得下式成立

$$\begin{cases} \Delta_x h(x,y) = 0 & \text{in } \Omega \\ h(x,y) = -v(x-y) & \text{on } \partial\Omega \end{cases}$$ （存在性要涉及到泛函分析）

由第二 Green 公式得 $\int_\Omega(u\Delta h - h\Delta u)dx = \int_{\partial\Omega}\left(u\frac{\partial h}{\partial n} - h\frac{\partial u}{\partial n}\right)dS$

$$\Rightarrow \int_\Omega -h\Delta u\,dx = \int_{\partial\Omega}\left(u\frac{\partial h}{\partial n} + v(x-y)\frac{\partial u}{\partial n}\right)dS.$$

$$\Rightarrow \int_{\partial\Omega}\left(u\frac{\partial h}{\partial n} + v(x-y)\frac{\partial u}{\partial n}\right)dS + \int_\Omega h\Delta u\,dx = 0$$

定义 $G = h+v$. 将上式与定理1中表达式相加即得

$$u(y) = \int_{\partial\Omega} u\frac{\partial G}{\partial n}\,dS + \int_\Omega G\Delta u\,dx \quad \cdots(**)$$

由上式可知若得到了 G 的表达式. 则通过 $u\big|_{\partial\Omega}$ 及 Δu 即可得到 u 的表达式. 其中 G 称为 Ω 的 Green 函数.

对一般区域的 Green 函数是非常难求的，下面只讨论 Ω 为球和上半平面的情形

① $\Omega = B_1(0) \subset \mathbb{R}^3$

由 $G(x,y)$ 满足 $\begin{cases} \Delta_x G(x,y) = 0 & \text{in } B_1(0) \setminus \{y\} \\ G(x,y) = 0 & \text{on } \partial B_1(0) \end{cases}$.

考虑点 y 关于 $\partial B_1(0)$ 的反演点 $\tilde{y} = \frac{y}{|y|^2}$.

由电像法知可设 $G = -\frac{1}{4\pi|x-y|} + \frac{k}{4\pi|x-\tilde{y}|}$

令 $x = \frac{y}{|y|} \in \partial B_1(0)$，代入得 $k = \frac{1}{|y|}$

从而 $G(x,y) = \frac{-1}{4\pi|x-y|} + \frac{1}{4\pi|x-\tilde{y}||y|}$

$$\frac{\partial G}{\partial x_i} = \frac{1}{4\pi}\frac{x_i-y_i}{|x-y|^3} - \frac{1}{4\pi}\frac{1}{|y|}\frac{x_i-\tilde{y}_i}{|x-\tilde{y}|^3}$$

$$= \frac{1}{4\pi}\frac{x_i-y_i}{|x-y|^3} - \frac{1}{4\pi}\frac{1}{|y|^3}\frac{x_i|y|^2-y_i}{|x-\tilde{y}|^3}$$

$$= \frac{1}{4\pi}\frac{x_i-y_i}{|x-y|^3} - \frac{1}{4\pi}\frac{x_i|y|^2-y_i}{|x-y|^3} \quad (\because |x-y| = |y|\,|x-\tilde{y}|)$$

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial x_i}\cdot x_i = \frac{1}{4\pi}\frac{x_i^2-x_iy_i}{|x-y|^3} - \frac{1}{4\pi}\frac{x_i^2|y|^2-y_ix_i}{|x-y|^3}$$

$$= \frac{1}{4\pi}\frac{1-|y|^2}{|x-y|^3}$$ 结合 $\Delta u = 0$，代入 u 的表达式

有 $$u(y) = \frac{1-|y|^2}{4\pi}\int_{\partial B_1(0)} \frac{\varphi(x)}{|x-y|^3}\,dS_x \quad (u|_{\partial\Omega} = \varphi)$$

此即为单位球上 Dirichlet 问题解的表达式

下面对该表达式进行验证.

(1) $\Delta u = 0$.

事实上，只需要验证 $\Delta_y\left(\frac{1-|y|^2}{|x-y|^3}\right) = 0$ 即可

易见 $(1-|y|^2)_i = -2y_i \qquad (1-|y|^2)_{ii} = -2$.

$$\left(|x-y|^{-3}\right)_i = -3\,|x-y|^{-5}(y_i - x_i)$$

$$\left(|x-y|^{-3}\right)_{ii} = -3|x-y|^{-5} + 15\,|x-y|^{-7}(y_i-x_i)^2$$

故 $$\left(\frac{1-|y|^2}{|x-y|^3}\right)_{ii} = \frac{-2}{|x-y|^3} + 2\cdot(-2y_i)\,\frac{-3(y_i-x_i)}{|x-y|^5}$$

$$+ (1-|y|^2)\left[-3|x-y|^{-5} + 15|x-y|^{-7}(y_i-x_i)^2\right]$$

$$\sum_{i=1}^{3}\left(\frac{1-|y|^2}{|x-y|^3}\right)_{ii} = \frac{-6}{|x-y|^3} + \frac{12(y_iy_i - x_iy_i)}{|x-y|^5}$$

$$+ (1-|y|^2)\left[-9|x-y|^{-5} + 15|x-y|^{-5}\right]$$

$$= -\frac{6}{|x-y|^3} + \frac{12|y|^2 - 12x_iy_i}{|x-y|^5} + \frac{6(1-|y|^2)}{|x-y|^5}$$

$$= \frac{-6+12x_iy_i-6|y|^2}{|x-y|^5} + \frac{12|y|^2-12x_iy_i}{|x-y|^5} + \frac{6-6|y|^2}{|x-y|^5} = 0$$

即 $\Delta\left(\dfrac{1-|y|^2}{|x-y|^3}\right) = 0$ 从而 $\Delta_y u = 0$.

(2) $$\int_{\partial B_1(0)} \frac{1-|y|^2}{4\pi}\frac{1}{|x-y|^3}\,dS_x = 1$$

可以直接计算验证上式. 也可以考虑如下问题的解

$$\begin{cases} \Delta u = 0 & \text{in } B_1(0) \\ u = 1 & \text{on } \partial B_1(0) \end{cases}$$ 显然其唯一解为 $u \equiv 1$.

从而由 $u\equiv 1$ 连续到边, 将其代入 Green 表示式得

$$\int_{\partial B_1(0)} \frac{1-|y|^2}{4\pi}\frac{1}{|x-y|^3}\,dS_x = 1$$

(3) $$\lim_{y\to x_0} u(y) = \varphi(x_0) \qquad (x_0\in\partial B_1(0))$$

由 $$u(y) - \varphi(x_0) \le \int_{\partial B_1(0)} \frac{1-|y|^2}{4\pi}\frac{|\varphi(x)-\varphi(x_0)|}{|x-y|^3}\,dS_x$$

$$= \frac{1-|y|^2}{4\pi}\int_{\partial B_1(0)\cap B(x_0,\delta)} \frac{|\varphi(x)-\varphi(x_0)|}{|x-y|^3}\,dS_x$$

$$+ \frac{1-|y|^2}{4\pi}\int_{\partial B_1(0)\setminus B(x_0,\delta)} \frac{|\varphi(x)-\varphi(x_0)|}{|x-y|^3}\,dS_x \triangleq A+B.$$

（其中 δ 是 s.t. 当 $|x-x_0|<\delta$ 时有 $|\varphi(x)-\varphi(x_0)|<\varepsilon$ 的数）

$$A \le \int_{\partial B_1(0)} \frac{1-|y|^2}{4\pi}\cdot\varepsilon\,dS_x = \varepsilon.$$

$\because$ $|x-x_0|\ge\delta$ 时 若 $|y-x_0|<\frac{\delta}{2}$，则有

$$|x_0-x|\le|x_0-y|+|y-x|\le|y-x|+\frac{\delta}{2}\le|y-x|+\frac{|x-x_0|}{2}$$

$\Rightarrow \frac{1}{2}|x_0-x|\le|y-x|$. 从而有

$$B\le\frac{1-|y|^2}{4\pi}\int_{\partial B_1(0)\setminus B(x_0,\delta)}\frac{2\|\varphi\|_{L^\infty}}{\left(\frac{|x_0-x|}{2}\right)^3}\,dS_x$$

$$\le C(1-|y|^2)\int_{\partial B_1(0)\setminus B(x_0,\delta)}\frac{1}{|x_0-x|^3}\,dS_x$$

$$\le C(1-|y|^2)\int_\delta^{+\infty}\frac{1}{r^3}\,r\,dr = \frac{C}{\delta}(1-|y|^2)$$

故 $y\to x_0$ 时 $|y|^2\to1$ 从而 $B\to0$.

再结合 ε 任意性知 $\lim\limits_{y\to x_0}u(y)=u(x_0)$

综上 (1), (3) 我们验证了所给出的表达式确实为原方程的解.

② 上半平面 $\mathbb{R}_+^3$ 情形. $\left(\mathbb{R}_+^3=\{(x_1,x_2,x_3)\mid x_3>0\}\right)$

同之前有 $\begin{cases}\Delta_x G(x,y)=0 & \text{in } \mathbb{R}_+^n\setminus\{y\}\\ G(x,y)=0 & \text{on } \partial\mathbb{R}_+^n\end{cases}$

考虑 y 关于 $\partial\mathbb{R}_+^n$ 的反射点 $\bar{y}=(y_1,y_2,-y_3)$

由电像法可令 $G = -\dfrac{1}{4\pi|x-y|} + \dfrac{1}{4\pi|x-\bar{y}|}$

易见 $y_3 = 0$ 时 $y = \bar{y} \Rightarrow G = 0$ on $\partial\mathbb{R}_+^n$

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial x_3} = \frac{-1}{4\pi}\left[\frac{x_3 - y_3}{|y-x|^n} - \frac{x_3+y_3}{|\bar{y}-x|^n}\right]$$

$$= \frac{y_3}{2\pi}\frac{1}{|x-y|^3}$$ 从而结合 $\Delta u = 0$ 代入表达式得

$$u(y) = \int_{\partial\mathbb{R}_+^3} \frac{y_3}{2\pi}\frac{\varphi(x)}{|x-y|^3}\,dS_x$$

下面验证之.(类似球形区域情形)

(1) $\Delta_y u = 0$

只要证 $\Delta_y\left(\dfrac{y_3}{|x-y|^3}\right) = 0$ 即可

$$\left(\frac{y_3}{|x-y|^3}\right)_{y_1y_1} = \frac{-3y_3}{|x-y|^5} + \frac{15(y_1-x_1)^2}{|x-y|^7}\cdot y_3$$

$$\left(\frac{y_3}{|x-y|^3}\right)_{y_2y_2} = -\frac{3y_3}{|x-y|^5} + \frac{15|y_2-x_2|^2}{|x-y|^7}\cdot y_3$$

$$\left(\frac{y_3}{|x-y|^3}\right)_{y_3y_3} = -\frac{3y_3}{|x-y|^5} + \frac{15(y_3-x_3)^2}{|x-y|^7}y_3 + 2\frac{-3(y_3-x_3)}{|x-y|^5}$$

三式相加 结合 $x_3 = 0$ 即得 $\Delta_y\left(\dfrac{y_3}{|x-y|^3}\right) = 0$

(2) 同球情形的(2)知 $\displaystyle\int_{\partial\mathbb{R}_+^3}\frac{y_3}{2\pi}\frac{1}{|x-y|^3}\,dS_x = 1$

(3) $\displaystyle\lim_{y\to x_0} u(y) = \varphi(x_0)$

$$|u(y) - \varphi(x_0)| = \left|\int_{\partial\mathbb{R}_+^3}\frac{y_3}{2\pi}\frac{\varphi(x)-\varphi(x_0)}{|x-y|^3}\,dS_x\right|$$

$$\le \frac{y_3}{2\pi}\left[\int_{\partial\mathbb{R}_+^3\cap B(x_0,\delta)}\frac{|\varphi(x)-\varphi(x_0)|}{|x-y|^3}\,dS_x\right.$$

$$\left.+\int_{\partial\mathbb{R}_+^3\setminus B(x_0,\delta)}\frac{|\varphi(x)-\varphi(x_0)|}{|x-y|^3}\,dS_x\right] \triangleq A + B$$

(当 $|x-x_0|<\delta$ 时 $|\varphi(x)-\varphi(x_0)|<\varepsilon$)

易见 $A \le \int_{\partial R_+^3 \cap B(x_0,\delta)} \frac{y_3}{2\pi} \frac{\varepsilon}{|x-y|^3} dS_x \le \varepsilon$

当 $|y-x_0| \le \frac{\delta}{2}$ 时，有 $|x-x_0| \le |x-y| + |y-x_0|$

$\le |x-y| + \frac{\delta}{2} \le |x-y| + \frac{|x-x_0|}{2} \Rightarrow |x-y| \ge \frac{1}{2}|x-x_0|$

故 $B \le \int_{\partial R_+^3 \setminus B(x_0,\delta)} \frac{2\|\varphi\|_{L^\infty}}{\left(\frac{1}{2}|x-x_0|\right)^3} \cdot \frac{y_3}{2\pi} dS_x$

$\le C y_3 \int_{\partial R_+^3 \setminus B(x_0,\delta)} \frac{dS_x}{|x-x_0|^3} \le C y_3 \int_\delta^{+\infty} \frac{1}{r^3} r dr$

$= C y_3 \left(-\frac{1}{r}\right)\Big|_\delta^{+\infty} = \frac{C y_3}{\delta} \to 0$ ($y \to 0$ 时)

故有 $|u(y)-\varphi(x_0)| \le \varepsilon$，从而 $\lim_{y\to x_0} u(y) = \varphi(x_0)$

这就验证了上半平面情形 $u(y)$ 表达式是正确的.

§14.2 调和函数的性质

上节我们得到了 $B_1(0) \subset \mathbb{R}^3$ 上的 Green 函数及

$$u(y) = \frac{1-|y|^2}{4\pi} \int_{\partial B(0,1)} \frac{\varphi(x)}{|x-y|^3} dS_x$$

易知 $B_R(0)$ 上的函数有如下表达式

$$u(y) = \frac{R^2-|y|^2}{4\pi R} \int_{\partial B(0,R)} \frac{\varphi(x)}{|x-y|^3} dS_x$$

令 $y=0 \Rightarrow u(0) = \frac{R^2}{4\pi R} \int_{\partial B(0,R)} \frac{\varphi(x)}{R^3} dS_x$

$$= \frac{1}{4\pi R^2} \int_{\partial B(0,R)} \varphi(x) dS_x \quad \cdots ①$$

上式称为平均值公式. 事实上，由①得

$$4\pi\rho^2 u(0) = \int_{\partial B(0,\rho)} u(x) dS_x$$

$$\Rightarrow \int_0^R 4\pi\rho^2 u(0)\, d\rho = \int_0^R \int_{\partial B(0,\rho)} u(x)\, dS_x\, d\rho$$

$$\Rightarrow u(0)\cdot \frac{4}{3}\pi R^3 = \int_{B_R(0)} u(x)\, dx$$

即 $$u(0) = \frac{1}{|B_R(0)|}\int_{B_R(0)} u(x)\, dx \quad \cdots\cdots ②$$

②式也称为平均值公式.

强极值原理：$\Delta u = 0$ in Ω. 则 $\max\limits_{\Omega} u \le \max\limits_{\partial\Omega} u$.

且若 $\exists x_0 \in \Omega$. s.t. $u(x_0)$ 取到最大值. 则 $u \equiv const$ in Ω

证明：定理的第一部分前面章节已证. 下面设 $x_0 \in \Omega$ 处

$u(x_0) = M = \max\limits_{\Omega} |u|$ $\because \Omega$ 为开集. $\therefore \exists B_r(x_0) \subset \Omega$.

由平均值公式知 $u(x_0) = ⨍_{B_r(x_0)} u(x)\, dx$

$$\triangleq \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(x)\, dx \le \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} M\, dx = M.$$

而 $u(x_0) = M$，故上式等号成立.

即 $\int_{B_r(x_0)} (u(x) - M)\, dx = 0$. $\Rightarrow u(x) = M$ in $B_r(x_0)$.

定义 $\Omega_M \triangleq \{x\in\Omega \mid u(x) = M\}$. 由上述性质知 Ω_M 为开集

($\because x_0 \in \Omega_M \Rightarrow B_r(x_0) \subset \Omega_M$) 再由 u 的连续性可知

Ω_M 为闭集. 而 Ω 为连通集. 从而知 $\Omega_M = \phi$ 或 $\Omega_M = \Omega$.

由假设 $\exists x_0 \in \Omega_M$. 故 $\Omega_M \ne \phi$. 从而 $\Omega_M = \Omega$.

即 $\forall x \in \Omega$ 均有 $u(x) = M$. 从而即 $u \equiv const$.

注：强极大值原理也可以利用 Hopf 引理来证明

而且 Hopf 引理的证明可以推广到一般线性椭圆方程

$Lu = 0$. 此时平均值公式便不再成立.

下面讨论逆平均值公式.

定理1: 设 $u \in C^0(\Omega)$, 且满足 $\forall B_R(x_0)$.

有 $u(y) = \frac{1}{\omega_n R^n}\int_{B_R(x_0)} u\,dx$. 则 $\Delta u = 0$.

证明: 由球上的 Poisson 公式得. $\forall$ 球 $B \subset\subset \Omega$.

$\exists$ B 内的调和函数 h, 在 ∂B 上 $h = u$.

令 $w \triangleq u - h$. 则有 $w \in C(\bar{B})$ 且 $w|_{\partial B} = 0$.

而 $\Delta h = 0$. 故 w 也具有平均值性质. 由强极值原理

故 $\max\limits_{\bar{B}} |w| = \max\limits_{\partial B} |w| = 0$. 从而 $u \equiv h$.

即 u 也为调和函数.

下面我们用两种方法来证明 Harnack 不等式.

定理2: u 为 $\Omega \subset \mathbb{R}^n$ 内的调和函数. $u \geq 0$

则 $\forall \Omega' \subset\subset \Omega$, $\exists$ $C(n)$. s.t.

$$\sup_{\Omega'} u \leq C(n) \inf_{\Omega'} u.$$

Step 1: 取 $y \in \Omega$, 选取 $R > 0$. s.t. $B_{4R}(y) \subset\subset \Omega$

$\forall x_1, x_2 \in B_R(y)$ 由平均值公式有

$$u(x_1) = \frac{1}{\omega_n R^n}\int_{B_R(x_1)} u(x)\,dx \leq \frac{1}{\omega_n R^n}\int_{B_{2R}(y)} u\,dx$$

$$u(x_2) = \frac{1}{\omega_n (3R)^n}\int_{B_{3R}(x_2)} u(x)\,dx \geq \frac{1}{\omega_n (3R)^n}\int_{B_{2R}(y)} u\,dx$$

故有 $u(x_1) \leq 3^n u(x_2)$ 从而得到 ($\because x_1, x_2$ 任意)

$$\sup_{B_R(y)} u \leq 3^n \inf_{B_R(y)} u$$

Step 2: 设 $\Omega' \subset\subset \Omega$. 则 $\exists x_1, x_2 \in \partial\Omega'$

使 $u(x_1) = \sup\limits_{\Omega'} u$ $\quad u(x_2) = \inf\limits_{\Omega'} u$

令 $\Gamma\subset\overline{\Omega'}$ 为连接 x_1. x_2 的弧. 选 R s.t.

$4R<\mathrm{dist}(\partial\Omega',\partial\Omega)$. 由有限覆盖定理知

$\overline{\Omega'}$ 被 N 个半径为 R 的球所覆盖. 故也覆盖 Γ.

对每一个球利用估计. 则得到 $u(x_1)\le 3^{nN}u(x_2)$.

上述证明中利用了平均值公式. 而对于一般流形上的 Laplace 方程其不再成立. 所以无法进行推广.

下面我们给出 Harnack 不等式的对数梯度估计证明.

其在流形上的推广可参考《微分几何讲义》丘成桐, 孙理察.

若 $\exists u(x_0)=0$ in $B_R(0)$. 则有 $u\equiv 0$. 结论得证.

下面我们假设 $u>0$ in $B_R(0)$. 令 $v=\ln u$.

则 $u=e^v$. $u_i=e^v v_i$ $u_{ii}=e^v v_i^2+e^v v_{ii}$

$$\Rightarrow \Delta u=e^v|\nabla v|^2+e^v\Delta v \Rightarrow \Delta v=-|\nabla v|^2.$$

定义 $\varphi=(R^2-|x|^2)^2|\nabla v|^2$. 易见 $\varphi=0$ on $\partial B_R(0)$

而 $\varphi\ge 0$ in $B_R(0)$. 故 $\exists x_0\in B_R(0)$ 取最大值

则有 $\varphi_i(x_0)=0$ 且 $\Delta\varphi(x_0)\le 0$

而 $\varphi_i=2(R^2-|x|^2)(-2x_i)|\nabla v|^2+(R^2-|x|^2)^2(|\nabla v|^2)_i$

在 x_0 处 $\varphi_i(x_0)=0\Rightarrow |\nabla v|^2_i(R^2-|x|^2)^2=4x_i(R^2-|x|^2)|\nabla v|^2$

$$\Rightarrow(|\nabla v|^2)_i=\frac{4x_i}{R^2-|x|^2}|\nabla v|^2 \quad \cdots ①$$

又 $\varphi_{ii}=[(R^2-|x|^2)^2]_{ii}|\nabla v|^2$

$+2((R^2-|x|^2)^2)_i(|\nabla v|^2)_i+(R^2-|x|^2)^2(|\nabla v|^2)_{ii}$

$\triangleq A+B+C$

下面先计算 B:

由①知 $B = 4(R^2-|x|^2)(-2x_i)(|\nabla v|^2)_i$

$= -8x_i \cdot 4x_i |\nabla v|^2 = -32|x|^2|\nabla v|^2$

再计算 C：由 $(|\nabla v|^2)_{ii} = (v_j^2)_{ii} = (2v_j v_{ji})_i$

$= 2v_{ji}^2 + 2v_j(\Delta v)_j = 2v_{ji}^2 - 2v_j(|\nabla v|^2)_j$

$\geq 2\sum_i v_{ii}^2 - 2v_j \dfrac{4x_j}{R^2-|x|^2}|\nabla v|^2$

$\geq \dfrac{2}{n}(\Delta v)^2 - \dfrac{8x_i v_i}{R^2-|x|^2}|\nabla v|^2$.

$C = (R^2-|x|^2)^2(|\nabla v|^2)_{ii} = \dfrac{2}{n}(R^2-|x|^2)^2|\nabla v|^4$

$-8x_i v_i(R^2-|x|^2)|\nabla v|^2$

最后计算 A. 由 $(R^2-|x|^2)^2_{ii} = [2(R^2-|x|^2)(-2x_i)]_i$

$= 8x_i^2 - 4(R^2-|x|^2)$ 知

$\Delta(R^2-|x|^2)^2 = 8|x|^2 - 4n(R^2-|x|^2)$ 代入有

$A = [(R^2-|x|^2)^2]_{ii}|\nabla v|^2 = [(8+4n)|x|^2 - 4nR^2]|\nabla v|^2$

综上所述 $0 \geq \Delta\varphi(x_0) = A+B+C$

$\geq [(8+4n)|x|^2 - 4nR^2]|\nabla v|^2 - 32|x|^2|\nabla v|^2$

$+\dfrac{2}{n}(R^2-|x|^2)^2|\nabla v|^4 - 8x_i v_i(R^2-|x|^2)|\nabla v|^2$

$\Rightarrow \dfrac{2}{n}|\nabla v|^4(R^2-|x|^2)^2 \leq 8x_i v_i|\nabla v|^2(R^2-|x|^2)$

$+4nR^2|\nabla v|^2 + (24-4n)|x|^2|\nabla v|^2$

$\Rightarrow \dfrac{2}{n}|\nabla v|^2(R^2-|x|^2)^2 \leq 8x_i v_i(R^2-|x|^2)$

$+(4n+24)R^2$

由基本不等式 $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ 知

$$8x_iv_i(R^2-|x|^2) \le \frac{1}{n}|\nabla v|^2(R^2-|x|^2)^2 + 4n|x|^2$$

故 $\frac{2}{n}|\nabla v|^2(R^2-|x|^2)^2 \le \frac{1}{n}|\nabla v|^2(R^2-|x|^2)^2$

$+ 4nR^2 + (4n+24)R^2$

$$\Rightarrow |\nabla v|^2(R^2-|x|^2)^2 \le (8n+24)R^2.$$

以上均在 x_0 处计算．上式即 $\varphi(x_0) \le \tilde{C}_n R^2$．

$\because x_0$ 处 φ 取最大值，从而 $\forall x \in B_R(0)$．

有 $|\nabla v|^2(R^2-|x|^2)^2 \le (8n+24)R^2$．

令 $|x| \le \frac{R}{2} \Rightarrow |\nabla v|(R^2-|x|^2) \le \sqrt{\tilde{C}_n}\, R$

$$\Rightarrow |\nabla v| \le \frac{B_n R}{R^2-|x|^2} \le \frac{B_n R}{\frac{3}{4}R^2} \le \frac{A_n}{R}.$$

故 $\forall x_1, x_2 \in B_{\frac{R}{2}}(0)$．$|v(x_1)-v(x_2)|$

$= \left|\int_0^1 \frac{d}{dt} v(tx_1+(1-t)x_2)\,dt\right|$

$= \left|(x_1-x_2)\int_0^1 \nabla v(tx_1+(1-t)x_2)\,dt\right|$

$\le |x_1-x_2| \int_0^1 \frac{A_n}{R}\,dt \le A_n$

从而 $\frac{u(x_1)}{u(x_2)} = e^{v(x_1)-v(x_2)} \le e^{A_n} = C_n$．

由 x_1、x_2 的任意性可知有 Harnack 不等式

$$\sup_{B_{\frac{R}{2}}(0)} u \le C_n \inf_{B_{\frac{R}{2}}(0)} u \quad \text{成立}.$$

同之前的方法可推广到 $\sup_{\Omega'} u \le C_n \inf_{\Omega'} u$ 其中 $\Omega' \subset\subset \Omega$．

下面证明导数的局部估计

定理：设 $\Delta u = 0$ in Ω，则 $\forall B_r(x_0) \subset\subset \Omega$

有 $|D^k u(x_0)| \leq \dfrac{C_k}{r^k} \|u\|_{L^\infty(B_r(x_0))}$

证明：令 $\eta = 1-|x|^2$．$\varphi = \eta^2 |\nabla u|^2 + \alpha u^2$．

其中 $\alpha > 0$ 待定．$\varphi_i = 2\eta\eta_i |\nabla u|^2 + \eta^2(|\nabla u|^2)_i + 2\alpha u u_i$

$$\varphi_{ii} = (\eta^2)_{ii} |\nabla u|^2 + 2(\eta^2)_i (|\nabla u|^2)_i + \eta^2 (|\nabla u|^2)_{ii} + \alpha(u^2)_{ii}$$

$$\Delta\varphi = \Delta(\eta^2)|\nabla u|^2 + 4\eta\eta_i(|\nabla u|^2)_i + \eta^2\Delta(|\nabla u|^2) + 2\alpha u_i^2$$

$$= [8|x|^2 - 4n(1-|x|^2)]|\nabla u|^2 + 4\eta\eta_i 2u_j u_{ji}$$

$$+ \eta^2 2|D^2u|^2 + 2\alpha|\nabla u|^2$$

$$\geq -4n|\nabla u|^2 + 8\eta\eta_i u_j u_{ij} + 2\eta^2 |D^2 u|^2 + 2\alpha|\nabla u|^2$$

$$\geq -4n|\nabla u|^2 - 2\eta^2 u_{ij}^2 - 8\eta_i^2 u_j^2 + 2\eta^2|D^2u|^2 + 2\alpha|\nabla u|^2$$

$$= (2\alpha - 4n - 8|\nabla\eta|^2)|\nabla u|^2 \geq (2\alpha - 4n - 32)|\nabla u|^2$$

则当 $\alpha \geq 2n+16$ 时有 $\Delta\varphi \geq 0$．

$$\Rightarrow \max_{B_{\frac{1}{2}}(0)} \eta^2|\nabla u|^2 \leq \max_{B_1(0)} \varphi \leq \max_{\partial B_1(0)} \varphi = \alpha \max_{\partial B_1(0)} u^2$$

$$\Rightarrow (1-|x|^2)^2 |\nabla u|^2 \leq \alpha \|u\|^2_{L^\infty(B_r(x_0))} \quad (|x| \leq \tfrac{1}{2})$$

$$\Rightarrow (1-|x|^2)|\nabla u| \leq \sqrt{\alpha}\|u\|_{L^\infty(B_r(x_0))} \quad (|x| \leq \tfrac{1}{2})$$

$$\Rightarrow |\nabla u| \leq C\|u\|_{L^\infty(B_r(x_0))} \quad (|x| \leq \tfrac{1}{2})$$

这就证明了 $k=1$ 时的结论．

从 $k=1$ 到一般的 k 是显然的，因为只要反复利用

$|Du(x_0)| \leq \dfrac{C}{r}\|u\|_{L^\infty(B_{\frac{r}{k}}(x_0))}$ 即可．

卷积与光滑子

设 $\Omega \subset \mathbb{R}^n$ 为开集，$\varepsilon > 0$.

定义 $\Omega_\varepsilon \triangleq \{x \in U \mid dist(x, \partial\Omega) > \varepsilon\}$.

定义 $\eta(x) \triangleq \begin{cases} C e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$

$\eta_\varepsilon(x) \triangleq \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$ 则 $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$

$supp\, \eta_\varepsilon \subset B(0, \varepsilon)$.

定义：若 f 为 Ω 上的局部可积函数

$f^\varepsilon \triangleq \eta_\varepsilon * f \quad in\ \Omega_\varepsilon$.

即 $f^\varepsilon(x) = \int_\Omega \eta_\varepsilon(x-y) f(y)\, dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y)\, dy$

光滑子具有下列性质.

(1) $f^\varepsilon \in C^\infty(\Omega)$

(2) 若 $f \in C(\Omega)$，则 $\{f^\varepsilon\}$ 关于 ε 在 Ω 内闭一致收敛到 f

证明：(1) $\forall x \in \Omega_\varepsilon$，$i \in \{1, 2 \cdots n\}$，$h$ 充分小

$x + he_i \in U_\varepsilon$，则有

$$\frac{f^\varepsilon(x+he_i) - f^\varepsilon(x)}{h} = \frac{1}{\varepsilon^n}\int_\Omega \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] f(y)\, dy$$

$$= \frac{1}{\varepsilon^n}\int_V \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] f(y)\, dy$$

$\forall V \subset\subset U$，$V$ 为开集.

$$\because \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] \rightrightarrows \frac{1}{\varepsilon}\eta_{x_i}\left(\frac{x-y}{\varepsilon}\right) \quad in\ V$$

$$f^\varepsilon_{x_i}(x) = \int_\Omega \eta_{\varepsilon, x_i}(x-y) f(y)\, dy$$

类似可证 $D^\alpha f_\varepsilon(x) = \int_\Omega D^\alpha \eta_\varepsilon(x-y) f(y)\, dy$

(2) $\because f \in C(\Omega)$ 设 K 为 Ω 的任一紧子集.

则 $f \in C(K)$. 从而 f 在 K 上一致连续.

$\forall \delta > 0$. $\exists \delta' > 0$. 当 $|y| < \delta'$ 时 $|f(x-y) - f(x)| < \delta$

又由定义知 $\mathrm{supp}\, \eta_\varepsilon = \overline{B(0.\varepsilon)}$ 从而 ε 充分小时

$\int_{|y| \geq \delta'} |\eta_\varepsilon(y)|\, dy = 0$. 故有下式

$$\sup_K |f^\varepsilon(x) - f(x)| = \sup_{x \in K} \left| \int [f(x-y) - f(x)]\, \eta_\varepsilon(y)\, dy \right|$$

$$\leq \sup_K |f(x-y) - f(x)| \int_{|y| < \delta'} |\eta_\varepsilon(y)|\, dy$$

$$+ \sup_K |f(x-y) - f(x)| \int_{|y| \geq \delta'} |\eta_\varepsilon(y)|\, dy < \delta$$

从而 $f^\varepsilon \rightrightarrows f$ in K. 由 K 任意性知结论成立

下面证明调和函数的正则性

定理: $\Delta u = 0$ in Ω. 则有 $u \in C^\infty(\Omega)$.

证明: $u^\varepsilon \triangleq u * \eta_\varepsilon \in C^\infty(\Omega_\varepsilon)$.

其中 $\Omega_\varepsilon \triangleq \{x \in \Omega \mid \mathrm{dist}(x.\partial\Omega) \geq \varepsilon\}$.

$$u^\varepsilon(x) = \int_\Omega \eta_\varepsilon(x-y)\, u(y)\, dy = \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\, dy$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B_r(x)} u\, dS \right) dr$$

$$= \frac{1}{\varepsilon^n} u(x) \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) n\omega_n r^{n-1}\, dr$$

$$= u(x) \int_{B_\varepsilon(0)} \eta_\varepsilon(y)\, dy = u(x)$$ 由 $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$

结合 ε 的任意性可知 $u \in C^\infty(\Omega)$

Liouville 定理.

全空间 $\mathbb{R}^n$ 上的有界调和函数为常数

证法一：由局部微商估计知 $\forall x_0 \in \mathbb{R}^n$.

$$|\nabla u(x_0)| \le \frac{n}{R}\|u\|_{L^\infty(B_R(x_0))} \le \frac{nM}{R}$$

其中 $M \triangleq \max\limits_{\mathbb{R}^n}|u|$ 令 $R \to +\infty$ 得 $\nabla u(x_0) = 0$.

由 x_0 任意性 $\nabla u \equiv 0$. 故 $u \equiv const$.

证法二：由 $u \le M$. 定义 $v \triangleq M - u \ge 0$.

且 $\Delta v = -\Delta u = 0$. 由对数梯度估计知

$$|\nabla \ln v| \le \frac{Cn}{R}$$ 令 $R \to +\infty$ 得 $\ln v \equiv const$.

即 $v \equiv const$. 从而 $u \equiv const$.

下面再证明两个关于调和函数的定理.

定理1：设 $\{u_n\}$ 为 Ω 内调和函数列. 且 $u_n(x)$ 一致收敛到 $u(x)$. 则 $u(x)$ 也调和

证明：$\forall B_r(x)$ 为 Ω 中的球. 由平均值公式知

$$u_n(x) = \fint_{B_r(x)} u_n(y)\,dy$$ 令 $n \to \infty$ 得

$$u(x) = \fint_{B_r(x)} u(y)\,dy$$ 由 $B_r(x)$ 的任意性

并利用逆平均值公式有 $\Delta u = 0$ in Ω.

定理2：若 $\{u_n\}$ 在 $\Omega \subset \mathbb{R}^n$ 内为单调递增的调和函数列

在 Ω 内一点 $y \in \Omega' \subset\subset \Omega$. $\{u_n(y)\}$ 收敛

则 u_n 在 Ω' 中一致收敛到一个调和函数

证明：$\forall \varepsilon > 0$. $\exists k \in \mathbb{N}^*$. 当 $m \ge n > k$ 时

$0 \le u_m(y) - u_n(y) < \varepsilon$. 由Harnack不等式

$$\sup_{\Omega'}(u_m - u_n) \le C\inf_{\Omega'}(u_m - u_n) \le C\big(u_m(y) - u_n(y)\big) < C\varepsilon.$$

从而 $\{u_n(x)\}$ 在 Ω' 中一致收敛到某函数 u.

由定理1 知 u 为调和函数.

奇点可去定理.

定理：设 $\Delta u = 0$ in $B_R(0)\setminus\{0\} \subset \mathbb{R}^n$ $(n \ge 3)$.

且有 $u(x) = o(|x|^{2-n})$ $\quad (|x| \to 0)$.

则可以补充定义 $u(0)$ s.t. u 在 $B_R(0)$ 中调和.

证明：取 v 满足 $\begin{cases} \Delta v = 0 & \text{in } B_R(0) \\ v = u & \text{on } \partial B_R(0) \end{cases}$.

$M \triangleq \max\limits_{\partial B_R(0)} |u|$. 又易知 $\pm M$ 为调和函数.

且有 $-M \le v \le M$. on $\partial B_R(0)$. 从而 $|v| \le M$ in $B_R(0)$.

定义 $w \triangleq u - v$. in $B_R(0)\setminus\{0\}$. $M_r \triangleq \max\limits_{\partial B_r(0)} |u|$.

则易知当 $x \in \partial B_R(0) \cup \partial B_r(0)$ 时下式成立.

$$-M_r \frac{r^{n-2}}{|x|^{n-2}} \le w(x) \le M_r \frac{r^{n-2}}{|x|^{n-2}}$$

由 $\pm M_r$ 为调和函数知 $|w| \le M_r \dfrac{r^{n-2}}{|x|^{n-2}}$ in $B_R(0)\setminus B_r(0)$

又 $M_r = \max\limits_{\partial B_r(0)} |v-u| \le \max\limits_{\partial B_r(0)} |v| + \max\limits_{\partial B_r(0)} |u|$

$\le M + \max\limits_{\partial B_r(0)} |u|$. 从而有

$$|w(x)| \le \frac{r^{n-2}}{|x|^{n-2}} M + \frac{1}{|x|^{n-2}}\Big(r^{n-2} \max_{\partial B_r(0)} |u|\Big)$$

令 $r \to 0^+$ 得 $w(x) = 0$ 由 x 任意性知

$u \equiv v$ in $B_R(0)\setminus\{0\}$ 补充定义 $u(0) = v(0)$

§14.3 Hopf 极值原理

定理1：设 $B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ $\Delta u \geq 0$ in $B_R(0)$.

设 $x_0 \in \partial B_R(0)$, $u \in C(\overline{B_R(0)})$. 且 $\forall x \in B_R(0)$.

有 $u(x_0) > u(x)$. 则有 $\left.\frac{\partial u}{\partial n}\right|_{x_0} > 0$.

证明：定义 $v(x) \triangleq e^{-\alpha|x|^2} - e^{-\alpha R^2}$

则 $v_i = -\alpha \cdot 2x_i e^{-\alpha|x|^2}$, $v_{ii} = -2\alpha e^{-\alpha|x|^2} + 4\alpha^2 e^{-\alpha|x|^2} x_i^2$.

$\Rightarrow \Delta v = -2\alpha n e^{-\alpha|x|^2} + 4\alpha^2 |x|^2 e^{-\alpha|x|^2}$

$= e^{-\alpha|x|^2}\left(4\alpha^2|x|^2 - 2\alpha n\right)$

定义 $A = B_R(0) \setminus \overline{B_{\frac{R}{2}}(0)}$. 则当 $\alpha > \frac{2n}{R^2}$ 时

$\Delta v \geq 0$ in A. 记 $w(x) \triangleq u(x) - u(x_0) + \varepsilon v(x)$.

则易见 $\begin{cases} \Delta w = \Delta u + \varepsilon \Delta v \geq 0 & \text{in } A \\ w|_{\partial B_R(0)} = u(x) - u(x_0) < 0 \end{cases}$,

而在 $\partial B_{\frac{R}{2}}(0)$ 上 $w(x) = u(x) - u(x_0) + \varepsilon\left(e^{-\frac{\alpha}{4}R^2} - e^{-\alpha R^2}\right)$

$\because u(x) - u(x_0) < 0$ in $B_R(0)$. $\therefore \exists C_0 > 0$.

s.t. $\forall x \in \overline{B_{\frac{R}{2}}(0)}$ 有 $u(x) - u(x_0) \leq -C_0$.

当 ε 充分小时 $u(x) - u(x_0) + \varepsilon\left(e^{-\frac{\alpha}{4}R^2} - e^{-\alpha R^2}\right) < 0$.

此即 $w(x) < 0$ on $\partial B_{\frac{R}{2}}(0)$.

综上所述. 我们有当 $\alpha > \frac{2n}{R^2}$ 且 ε 充分小时

$\begin{cases} \Delta w(x) \geq 0 & \text{in } A \\ w < 0 & \text{on } \partial A \end{cases}$ 从而 $w(x) \leq 0$ in A

又易见 $w(x_0) = 0$. 从而结合 $w \leq 0$ 知

$\frac{\partial w}{\partial n}\Big|_{x_0} \geq 0$. 即 $\frac{\partial u}{\partial n}\Big|_{x_0} \geq -\varepsilon \frac{\partial v}{\partial n}\Big|_{x_0}$

而 $\frac{\partial v}{\partial n} = -2\alpha e^{-\alpha r^2}$. 从而有

$\frac{\partial u}{\partial n}\Big|_{x_0} \geq 2\varepsilon\alpha e^{-\alpha r^2} > 0$. 这就证明了原定理.

下面利用 Hopf 引理证明强极大值原理

定理 2: 设 $\Delta u \geq 0$ in Ω. 则 u 的最大值只能在边界上达到 (即若 $\exists\, x_0 \in \Omega$, s.t. $u(x_0) = M = \max\limits_{\Omega} u$. 则必有 $u \equiv M$)

证明: 记 $M \triangleq \max\limits_{\Omega} u$ 且 $C \triangleq \{x \in \Omega \mid u(x) = M\}$.

$V \triangleq \{x \in \Omega \mid u(x) < M\}$ 取 $y \in V$ (若 $V \neq \phi$)

s.t. $\mathrm{dist}(y, C) < \mathrm{dist}(y, \partial\Omega)$

记 B 为中心在 y 且落在 V 内的最大的球

则 $\exists\, x_0 \in C$. 且 $x_0 \in \partial B$. 从而 V 在 x_0 处有一个内切球 B .

对球 B 应用 Hopf 引理可得 $\frac{\partial u}{\partial n}\Big|_{x_0} > 0$.

但 $x_0 \in C \Rightarrow u(x_0) = M \Rightarrow Du(x_0) = 0$

故矛盾. 从而 y 并不存在. 即 $V = \phi$ 或 $V = \Omega$.

从而证明了强极值原理

定义: 称一个区域在 x 处有内球条件是指在 x 点处有内切球. 例如上面定理中的 B 为内球.

定理 3: 考虑方程 $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$.

Ω 在每点有内球条件. 则 $u \equiv$ const .

证明：由强极值原理知 若 u 不恒为常数

则 u 必在边界处取极大值．设 $u(x_0)=\max\limits_{\Omega} u$．

由 Hopf 引理知 $\left.\frac{\partial u}{\partial n}\right|_{x_0}>0$．从而矛盾．故 $u\equiv const$．

注：Hopf 引理对于一般的椭圆方程

$a^{ij}u_{ij}+b^{i}u_{i}+cu\geq 0$（其中 $c\leq 0$）也是成立的．

从而由此可导出强极值原理．然而对于一般线性椭圆方程而言平均值公式便不再成立．无法用此种方法来得到强极值原理．从而 Hopf 引理比平均值公式更加具有普适性．详细的证明参考《Partial Differential Equations》L.C. Evans．

本书的 6.4 节完全且细致的讨论了上述问题，并且给出了 $a^{ij}u_{ij}+b^{i}u_{i}+cu=0$ 的 Harnack 不等式

（同样无法利用平均值公式）．

第十五章 Bessel 函数

在第七章中我们要求解以下方程

$$x^2y'' + xy' + (x^2-n^2)y = 0 . \quad (n \geq 0)$$

设 $y = \sum_{k=0}^{\infty} c_k \rho^{k+\rho} \quad (c_0 \neq 0)$ 代入后得

$$\sum_{k=0}^{\infty} \left[(\rho+n+k)(\rho-n+k)c_k + c_{k+2}\right] x^{k+\rho} = 0 .$$

$$\Rightarrow (\rho+n+k)(\rho-n+k)\, c_k + c_{k-2} = 0 . \quad (k=0.1.2\cdots)$$

$\because c_{-1} = c_{-2} = 0$. 且 $c_0 \neq 0$. 故 $(\rho+n)(\rho-n) = 0$.

从而得到 $\rho_1 = n$, $\rho_2 = -n$. 若 $\rho = \rho_1 = n$. 则

$$(2n+k)k\,c_k + c_{k-2} = 0 . \quad (k=1,2\cdots)$$

从而可以依次确定 c_k 如下 .

$k=1$ 时 $(2n+1)c_1 = 0 \Rightarrow c_1 = 0 \Rightarrow c_{2k+1} = 0$.

$k=2, 4. 6 \Rightarrow c_2 = -\dfrac{1}{2^2(n+1)} c_0$

$$c_4 = \frac{1}{2^4(n+1)(n+2)\,2!} c_0 \quad \cdots \quad c_{2k} = \frac{(-1)^k}{2^{2k}(n+1)\cdots(n+k)\,k!} c_0 .$$

利用 Γ 函数 . $c_{2k} = \dfrac{(-1)^k}{\Gamma(n+k+1)\Gamma(k+1)} \dfrac{1}{2^{2k+n}}$

$$y = J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(n+k+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+n}$$

$J_n(x)$ 称为第一类 Bessel 函数

对于 $\rho = \rho_2 = -n$. 有 $k(k-2n)c_k + c_{k-2} = 0$

(1) $2n \notin \mathbb{Z}$. 此时 $k(k-2n) \neq 0$.

取 $c_0 = \dfrac{1}{2^{-n}\Gamma(-n+1)}$

$$y = J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(-n+k+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k-n} \quad \cdots (*)$$

上式称为第二类 Bessel 函数

(2) $2n = N \in \mathbb{Z}$.

① $2n = 2s+1$. 由递推式知. 当 k 为偶数时

$k(k-2n) \neq 0$. 同(1)可确定 C_k. 当 k 为奇数时

若 $k < 2s+1$. 则 C_k $(k \geq 1)$ 的系数 $k(k-2n) \neq 0$.

从而 $C_1 = \cdots = C_{2s-1} = 0$.

若 $k \geq 2s+1$. 则 C_{2s+1} 系数为 0. 且有

$(2s+1)(-2n+2s+1)C_{2s+1} = 0$, $(2s+3)(-2n+2s+3)C_{2s+3} + C_{2s+1} = 0$

$\cdots$

因此. 只要令 $C_{2s+1} = 0$. 则仍有 $C_{2s+3} = C_{2s+5} = \cdots = 0$.

所以此种情形 $J_{-n}(x)$ 表达式与 $(*)$ 式相同.

② $2n$ 为偶数. 故 $C_2 \neq 0$, $C_4 \neq 0 \cdots$

且 $2n(2n-2n)C_{2n} + C_{2n-2} = 0 \Rightarrow C_{2n-2} = 0$.

从而不会有广义幂级数解. 为此取 $\alpha \neq n$.

$J_\alpha(x)$, $J_{-\alpha}(x)$ 都有意义. $y_\alpha(x) \triangleq \dfrac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi}$

易见 y_α 为原方程的解. $\alpha \to 0$ 时为 $\frac{0}{0}$ 型不定式.

$$Y_n(x) = \lim_{\alpha\to n} y_\alpha(x) = \lim_{\alpha \to n} \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi}$$

$Y_n(x)$ 称为 Neumann 函数或第二类 Bessel 函数

微分方程I习题课

第一次习题课

1. 求解 $(y^2-6x)\frac{dy}{dx}+2y=0$.

解法一：$\frac{dx}{dy}=\frac{6x-y^2}{2y}=-\frac{y}{2}+\frac{3x}{y}$.

$$x=e^{\int\frac{3}{y}dy}\left(c+\int-\frac{y}{2}e^{-\int\frac{3}{y}dy}dy\right)$$

$$=y^3\left(c+\int-\frac{1}{2y^2}dy\right)=cy^3+\frac{y^2}{2}$$

加上特解 $y\equiv 0$

解法二：$2y\,dx+(y^2-6x)\,dy=0$.

$P=2y$. $Q=y^2-6x$.

$$\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\frac{1}{2y}(-8)=-\frac{4}{y}$$

$$\mu(y)=e^{-\int\frac{4}{y}dy}=\frac{1}{y^4}$$

$$\frac{2}{y^3}dx+\left(\frac{1}{y^2}-\frac{6x}{y^4}\right)dy=0$$

$$\Rightarrow \frac{2x}{y^3}-\frac{1}{y}=C$$

$$\Rightarrow x=cy^3+\frac{y^2}{2}$$ $y\equiv 0$ 为特解.

2. 求解 $2x^4y\frac{dy}{dx}+y^4=4x^6$.

解：令 $y=z^m$. 有

$$2mz^{2m-1}x^4\frac{dz}{dx}+z^{4m}=4x^6.$$

$m=\frac{3}{2}$ 时为齐次. 从而令 $y=z^{\frac{3}{2}}$

有 $3x^4z^2\frac{dz}{dx}+z^6=4x^6$

$\Rightarrow 3\left(\frac{z}{x}\right)^2\frac{dz}{dx}+\left(\frac{z}{x}\right)^6=4$. 令 $u=\frac{z}{x}$.

$3u^2\left(x\frac{du}{dx}+u\right)+u^6=4$.

$\Rightarrow \frac{dx}{x}=-\frac{d(u^3)}{(u^3+4)(u^3-1)}$

$\Rightarrow \frac{1}{5}\ln\frac{|u^3-1|}{|u^3+4|}=-\ln|x|+c$.

$\Rightarrow \frac{u^3+4}{u^3-1}=Cx^5$ 代入得 $\frac{y^2+4x^3}{y^2-x^3}=Cx^5$

特解为 $y^2=x^3$

3. $\frac{dy}{dx}+p(x)y=q(x)$ --- $(*)$

$p(x)$. $q(x)$ 为以 $w>0$ 为周期的连续函数. 证明

(1) 若 $q\equiv 0$. 则 $(*)$ 的任一非零解以 w 为周期.

当且仅当 $\bar{p}\triangleq\frac{1}{w}\int_0^w p(x)\,dx=0$.

(2) 若 q 不恒为 0. 则 $(*)$ 有唯一周期解 $\Leftrightarrow \bar{p}\neq 0$.

试求出此解

解: (1) $y=Ce^{-\int p(x)dx}\quad (C\neq 0)$.

$y(x)=y(x+w) \Leftrightarrow e^{\int_x^{x+w}p(t)dt}=1$

$\Leftrightarrow \int_x^{x+w}p(t)\,dt=0 \Leftrightarrow \int_0^w p(t)\,dt=0 \Leftrightarrow \bar{p}=0$.

(2) $y=Ce^{-\int_0^x p(s)ds}+\int_0^x q(s)e^{-\int_s^x p(t)dt}ds$ ... $(**)$

令 $u(x)=y(x+w)-y(x)$. 则 $u'+p(x)u=0$

即 $u=Ce^{-\int_0^x p(t)dt}$ 从而 $u\equiv 0$ 或 u 恒不为 0.

从而 $y(x+w)\equiv y(x) \Leftrightarrow y(w)=y(0)$

$\Leftrightarrow Ce^{-\int_0^w p(s)ds}+\int_0^w q(s)e^{-\int_s^w p(t)dt}ds=C$

(*) 有唯一周期解 $\Longleftrightarrow$ $e^{-\int_0^\omega p(s)ds} \neq 1$

即 $\int_0^\omega p(s)\,ds \neq 0$ 即 $\bar{p} \neq 0$

将 $C = \dfrac{1}{1-e^{-\int_0^\omega p(s)ds}} \int_0^\omega q(s) e^{-\int_s^\omega p(t)dt} ds$

代入 (**) 即求出此解.

4. $f(x,y)$ 为关于 y 递减的连续函数 则初值问题

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

在 $x \geq x_0$ 的解唯一. 而在 $x \leq x_0$ 不一定唯一

解: 设 $y_1(x)$. $y_2(x)$ 为满足题意的两解. 且 $\exists x_1 > x_0$.

s.t. $y_1(x_1) > y_2(x_1)$. 记 $\xi \triangleq \sup\{x_0 \leq x < x_1 \mid y_1(x) = y_2(x)\}$.

则 $\forall x \in (\xi, x_1)$ 有 $y_1(x) > y_2(x)$

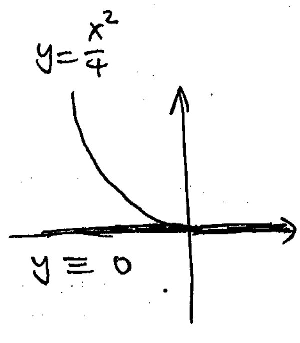
记 $r(x) \triangleq y_1(x) - y_2(x)$. $r'(x) = \dfrac{dy_1}{dx} - \dfrac{dy_2}{dx}$

$= f(x,y_1) - f(x,y_2) \leq 0$. 而 $r(\xi) = 0$

得 $r(x) \leq r(\xi) = 0$ in (ξ, x_1) 与 $y_1(x) > y_2(x)$ 矛盾

在左侧则不一定唯一.

例如 $\begin{cases} \dfrac{dy}{dx} = -|y|^{\frac{1}{2}} & x \geq 0 \\ y(0) = 0. \end{cases}$



有 $y = \begin{cases} \dfrac{x^2}{4} & x \leq 0 \\ 0 & x > 0 \end{cases}$ 即为其解

又 $y \equiv 0$ 也为其解. 从而在左侧有两个解.

Arzela－Ascoli 引理

有界闭区间 I 上 $\{f_n(x)\}$ 满足一致有界且等度连续

则 $\exists \{f_{n_k}(x)\}$ 在 I 上一致收敛.

证明：取 $I=[a,b]$ 中全体有理数 $\{r_1, r_2 \cdots r_n \cdots\}$

$\{f_n(r_1)\}$ 为有界数列. 由 B－W 定理知有函数序列

$f_{n_1}^{(1)}(x) \quad f_{n_2}^{(1)}(x) \cdots\cdots \quad f_{n_k}^{(1)}(x) \cdots$ 在 r_1 收敛.

同样由 $\{f_{n_k}^{(1)}(r_2)\}$ 为有界数列. 可从中取出子列收敛.

即 $f_{n_1}^{(2)}(x) \quad f_{n_2}^{(2)}(x) \cdots \quad f_{n_k}^{(2)}(x) \cdots$ 在 r_2 收敛.

以此类推. 我们有

$$\begin{array}{ccccl}
f_{n_1}^{(1)}(x) & f_{n_2}^{(1)}(x) & \cdots & f_{n_k}^{(1)}(x) & \text{在 } r_1 \text{ 收敛} \\
f_{n_1}^{(2)}(x) & f_{n_2}^{(2)}(x) & \cdots & f_{n_k}^{(2)}(x) & \text{在 } r_2 \text{ 收敛} \\
\vdots & \vdots & \ddots & \vdots & \\
f_{n_1}^{(s)}(x) & f_{n_2}^{(s)}(x) & \cdots & f_{n_k}^{(s)}(x) & \text{在 } r_s \text{ 收敛} \\
\vdots & \vdots & & \vdots &
\end{array}$$

其中每一行为上一行子列.

现取对角线序列 $f_{n_1}^{(1)}(x) \cdots f_{n_k}^{(k)}(x) \cdots$

在 r_s 处收敛 $(s\in N)$. 下证其在 I 上一致收敛.

$\because f_n$ 等度连续 $\therefore \forall \varepsilon>0.\ \exists \delta>0.\ |x_1-x_2|<\delta$ 时

$|f_n(x_1)-f_n(x_2)|<\varepsilon$. 又 $\because \bigcup\limits_{x\in I}\left(x-\frac{\delta}{2},\ x+\frac{\delta}{2}\right)$

为 I 的开覆盖. $\therefore \exists$ 有限个开区间 $\bigcup\limits_{j=1}^{n}\left(x_j-\frac{\delta}{2},\ x_j+\frac{\delta}{2}\right)\supset I$

取 $q_j\in \mathbb{Q}\cap\left(x_j-\frac{\delta}{2},\ x_j+\frac{\delta}{2}\right)\ (j=1.2\cdots n)$

由 $\{f_{n_k}^{(k)}(q_j)\}$ 收敛. $\forall\varepsilon>0.\ \exists N\in N^*.\ k>N$ 时

$\left|f_{n_k}^{(k)}(q_j)-f_{n_l}^{(l)}(q_j)\right|<\varepsilon \qquad (j=1.2\cdots n)$

$\forall x\in[a.b].\ \exists p\in\{1.2\cdots n\}.\ x\in\left(x_p-\frac{\delta}{2},\ x_p+\frac{\delta}{2}\right)$

又 $q_p \in (x_p - \frac{\delta}{2}, x_p + \frac{\delta}{2})$ $\therefore |x - q_p| < \delta$

$\therefore |f_n(x) - f_n(q_p)| < \varepsilon \quad \forall n \in N$

故 $l, k > N$ 时有

$$|f_{n_l}^{(l)}(x) - f_{n_k}^{(k)}(x)| \le |f_{n_l}^{(l)}(x) - f_{n_l}^{(l)}(q_p)|$$

$$+ |f_{n_l}^{(l)}(q_p) - f_{n_k}^{(k)}(q_p)| + |f_{n_k}^{(k)}(q_p) - f_{n_k}^{(k)}(x)|$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

由柯西收敛准则 $f_{n_k}^{(k)}(x) \rightrightarrows f(x) \quad (x \in I)$

压缩映射原理

定义 $C[a,b]$ 为 $[a,b]$ 上全体连续函数构成的空间

f, g 距离 $\rho(f,g) \triangleq \max\limits_{x \in [a,b]} |f(x) - g(x)|$

易见 (1) $\rho(f,g) \geq 0$

(2) $\rho(f,g) = \rho(g,f)$

(3) $\rho(f,h) \le \rho(f,g) + \rho(g,h)$

映射 T 称为压缩映射，如果 $\exists \theta \in (0,1)$

s.t. $\rho(Tf, Tg) \le \theta \rho(f,g)$

定理：设 T 为 $C[a,b]$ 上的压缩映射，则存在唯一 $x(t) \in C[a,b]$ s.t. $Tx = x$

证明：任取 $x_0(t) \in C[a,b]$，$x_{n+1} = Tx_n$.

$$\rho(x_{n+1}, x_n) \le \theta\rho(x_n, x_{n-1}) \le \cdots \le \theta^n \rho(x_1, x_0)$$

$$\rho(x_{n+p}, x_n) \le \sum_{i=0}^{p-1} \rho(x_{n+i+1}, x_{n+i}) \le \sum_{i=0}^{p-1} \theta^{n+i} \rho(x_1, x_0)$$

$$\le \rho(x_1, x_0) \sum_{i=n}^{\infty} \theta^i = \frac{\theta^n}{1-\theta} \rho(x_1, x_0)$$

即 $\lim\limits_{n \to \infty} \rho(x_{n+p}, x_n) = 0$

即 $x_n(t) \rightrightarrows x(t)$. 由 $x_{n+1} = Tx_n$

$\Rightarrow \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} T x_n \Rightarrow x = Tx$.

即存在不动点. 下证唯一性.

设 $\exists x'$ s.t. $Tx' = x'$.

则有 $\rho(x', x) = \rho(Tx', Tx) \leq \theta \rho(x', x)$

$\because 0 < \theta < 1$; $\therefore \rho(x', x) = 0$.

即 $x' = x$. 从而证明了唯一性.

Picard 存在唯一性定理 (用压缩映射证明)

证明: $h \triangleq \min\{a, \frac{b}{m}, \frac{1}{L}\}$.

在 $C[x_0-h, x_0+h]$ 上定义 $Ty \triangleq y_0 + \int_{x_0}^{x} f(x, y)\,dx$

由 $|Ty_1(x) - Ty_2(x)| = \left|\int_{x_0}^{x} f(x, y_1) - f(x, y_2)\,dx\right|$

$\leq \int_{x_0}^{x} |f(x, y_1) - f(x, y_2)|\,dx$

$\leq L \int_{x_0}^{x} |y_1 - y_2|\,dx \leq Lh\,\rho(y_1, y_2) \quad \forall x \in [x_0-h, x_0+h]$

故 $\rho(Ty_1, Ty_2) \leq Lh\,\rho(y_1, y_2) \quad Lh < 1$.

$\therefore T$ 在 $C[x_0-h, x_0+h]$ 上为压缩映射.

故存在唯一 y s.t. $Ty = y$ 即有

$y = y_0 + \int_{x_0}^{x} f(x, y(x))\,dx$ 成立

从而原方程解在局部存在且唯一.

第二次习题课

例1. $\begin{cases} \Delta u = -2 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ $\quad \Omega = \{ x_1^2 + \frac{x_2^2}{a^2} < 1 \}$.

解：由于 $\Delta \left(\frac{a^2 x_1^2 + x_2^2}{a^2+1} \right) = 2$.

令 $V = u + \frac{a^2 x_1^2 + x_2^2}{a^2+1}$

则有 $\begin{cases} \Delta V = \Delta u + 2 = 0 & \text{in } \Omega. \\ V = \frac{a^2 x_1^2 + x_2^2}{a^2+1} = \frac{a^2}{a^2+1} & \text{on } \partial\Omega. \end{cases}$

由极值原理知 $V \equiv \frac{a^2}{a^2+1}$ in Ω.

从而 $u = \frac{a^2 - a^2 x_1^2 - a^2 x_2^2}{a^2+1}$

例2. $\begin{cases} \Delta u = -2 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ $\quad \Omega = \{ x_1^2 + x_2^2 \le 1 \}$

(1) 解方程：在例1中令 $a=1$ 得 $u = \frac{1 - x_1^2 - x_2^2}{2}$.

(2) 解唯一性：只要证 $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \Rightarrow u \equiv 0$ 即可.

而 $\Delta u = 0 \Rightarrow 0 = \int_\Omega u \Delta u = \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds - \int_\Omega |\nabla u|^2$

$= -\int_\Omega |\nabla u|^2 \Rightarrow |\nabla u| \equiv 0 \Rightarrow u \equiv C \Rightarrow u \equiv 0$.

(3) Bernstein 技巧：$\varphi \triangleq |\nabla u|^2 + 2u$.

则 $\varphi_i = 2 u_k u_{ki} + 2 u_i$.

$\varphi_{ii} = 2 u_{ki}^2 + 2 u_k u_{kii} + 2 u_{ii}$.

$\Delta \varphi = 2 |D^2 u|^2 + 2 \nabla u \cdot \nabla(\Delta u) + 2 \Delta u$

$$\geq 2(u_{11}^2+u_{22}^2)+2\Delta u$$

$$\geq 2\frac{(u_{11}+u_{22})^2}{2}+2\Delta u = 4+2\times(-2)=0.$$

从而 $\Delta\varphi \geq 0$ 故 $\max\limits_{\Omega}\varphi = \max\limits_{\partial\Omega}\varphi$.

例3：设 $(x_1, x_2)\in B_1(0)\subset\mathbb{R}^2$.

(1) 若 $u\in C^\infty(\overline{B_1(0)})$ 满足 $\Delta u=0$. 证 $\sup\limits_{B_1(0)}|\nabla u|\leq \sup\limits_{\partial B_1(0)}|\nabla u|$.

(2) 若 $u\in C^\infty(\overline{B_1(0)})$ 是 $\Delta u+2u=0$ 的解.

证 $\exists\, c_1>0$, s.t. $\sup\limits_{B_1(0)}|\nabla u|\leq c_1\left(\sup\limits_{\partial B_1(0)}|\nabla u|+\sup\limits_{B_1(0)}|u|\right)$

解：(1) $\Delta(|Du|^2) = 2u_{ki}^2+2u_k u_{kii} = 2u_{ki}^2\geq 0$.

$$\therefore \sup_{\overline{B_1(0)}}|Du|^2 \leq \sup_{\partial B_1(0)}|Du|^2.$$

即 $\sup\limits_{B_1(0)}|\nabla u|\leq \sup\limits_{\partial B_1(0)}|\nabla u|$.

(2) $\Delta\left(|Du|^2+2u^2+\beta\frac{|x|^2}{4}\right)$

$$=2|D^2u|^2+2\nabla u\cdot(-2)\nabla u+4|Du|^2-8u^2+\beta.$$

$$=2|D^2u|^2-8u^2+\beta.$$ 取 $\beta = 8\sup\limits_{B_1(0)}|u|^2$.

则上式 ≥ 0. 从而有 $\sup\limits_{B_1(0)}\left(|Du|^2+2u^2+\beta\frac{|x|^2}{4}\right)$

$$\leq \sup_{\partial B_1(0)}\left(|Du|^2+2u^2+\beta\cdot\frac{|x|^2}{4}\right)$$

$$\leq \sup_{\partial B_1(0)}|Du|^2+2\sup_{\partial B_1(0)}u^2+\frac{\beta}{4}$$

$$\leq 4\sup_{\partial B_1(0)}|u|^2+\sup_{\partial B_1(0)}|Du|^2$$

$$\leq\left(\sup_{\partial B_1(0)}|Du|+2\sup_{\partial B_1(0)}|u|\right)^2$$

于是 $\sup\limits_{B_1(0)} |Du|^2 \le \left(\sup\limits_{\partial B_1(0)} |Du| + 2\sup\limits_{\partial B_1(0)} |u|\right)^2$

即 $\sup\limits_{B_1(0)} |Du| \le \sup\limits_{\partial B_1(0)} |Du| + 2\sup\limits_{\partial B_1(0)} |u|$.

从而可取 $C_1 = 2$ 、

例4: $u(t), v(t) \in C[a,b]$. $v(t) \ge 0$.

$u(t) \le u_0 + \int_a^t u(s)v(s)\,ds$

利用逐次逼近法证明 $u(t) \le u_0 e^{\int_a^t v(s)\,ds}$.

证明: 定义 $u_0(t) = u_0$.

$u_1(t) = u_0 + \int_a^t u_0 v(s)\,ds = u_0\left[1 + \int_a^t v(s)\,ds\right]$

....

$u_n(t) = u_0 + \int_a^t u_{n-1}(s)\,v(s)\,ds$

我们有 $u_n \in C[a,b]$. 且易见

$u_2(t) = u_0 + \int_a^t u_1(s)\,v(s)\,dt$

$= u_0 + \int_a^t \left[u_0 + u_0\int_a^s v(\xi)\,d\xi\right] v(s)\,ds$

$\le u_0 + u_0\int_a^t v(s)\,ds + \frac{u_0}{2!}\left(\int_a^t v(s)\,ds\right)^2$

一般地有 (可归纳证明)

$u_n(t) \le u_0\left[1 + \int_a^t v(s)\,ds + \cdots + \frac{1}{n!}\left(\int_a^t v(s)\,ds\right)^n\right] \cdots (*)$

事实上: 设 $u_{n-1}(t) \le u_0\left[1 + \int_a^t v(s)\,ds + \cdots + \frac{1}{(n-1)!}\left(\int_a^t v(s)\,ds\right)^{n-1}\right]$

则有 $u_n(t) \le u_0 + u_0\left(\int_a^t \left[1 + \int_a^s v(\xi)\,d\xi + \cdots + \frac{1}{(n-1)!}\left(\int_a^s v(\xi)\,d\xi\right)^{n-1}\right] v(s)\,ds\right.$

$\le u_0 + u_0\int_a^t v(s)\,ds + \cdots + \frac{u_0}{n!}\left(\int_a^t v(s)\,ds\right)^n$.

记 $M \triangleq \max\limits_{s\in[a,b]} u(s)$.

$$u(t)-u_0 \le \int_a^t u(s)v(s)ds \le M\int_a^t v(s)ds$$

$$u(t)-u_1(t) \le \int_a^t [u(s)-u_0]v(s)ds$$

$$\le \int_a^t M\left(\int_a^s v(\xi)d\xi\right)v(s)ds \le \frac{M}{2!}\left(\int_a^t v(s)ds\right)^2$$

设 $u(t)-u_{n-1}(t) \le \dfrac{M}{n!}\left(\int_a^t v(\xi)d\xi\right)^n$

则有 $u(t)-u_n(t) \le \int_a^t [u(s)-u_{n-1}(s)]v(s)ds$

$$\le \frac{M}{n!}\int_a^t\left(\int_a^s v(\xi)d\xi\right)v(s)ds$$

$$\le \frac{M}{(n+1)!}\left(\int_a^t v(\xi)d\xi\right)^{n+1}$$

从而 $\forall n\in N$, 有 $u(t)-u_n(t) \le \dfrac{M\left(\int_a^b v(\xi)d\xi\right)^{n+1}}{(n+1)!}$

$$\therefore u(t) \le u_n(t) + \frac{M}{(n+1)!}\left(\int_a^b v(\xi)d\xi\right)^{n+1}$$

结合(*)式知 $u(t) \le \lim\limits_{n\to\infty} u_n(t) \overset{(*)}{\le} u_0 e^{\int_a^t v(s)ds}$

例5: 求解方程: $y^2(y'-1)=(2-y')^2$

解: 令 $2-y'=yt$, 代入得

$1-yt=t^2$, 即 $y=\dfrac{1-t^2}{t}$.

$y'=2-yt=1+t^2$. 记 $p=y'$

由 $dy=d\left(\dfrac{1-t^2}{t}\right)=\left(-1-\dfrac{1}{t^2}\right)dt$

$dx=\dfrac{1}{p}dy=-\dfrac{1}{t^2}dt$ 从而 $x=\dfrac{1}{t}+C$

$y = \frac{1}{t} - t = x - c - \frac{1}{x-c}$ 为通解.

例6：求解方程 $x = a \cdot t \cdot \frac{dx}{dt} + (\frac{dx}{dt})^2$

解：$p \triangleq \frac{dx}{dt}$. 则 $x = atp + p^2$.

$\Rightarrow p = ap + at\frac{dp}{dt} + 2p\frac{dp}{dt}$. （$a \neq 1$ 时）.

$\Rightarrow \frac{dt}{dp} = \frac{at+2p}{(1-a)p} = \frac{at}{(1-a)p} + \frac{2}{1-a}$.

为一阶线性方程，可解之得

$t = cp^{\frac{a}{1-a}} + \frac{2}{1-2a}p$

$\Rightarrow x = atp + p^2 = cap^{\frac{1}{1-a}} + \frac{1}{1-2a}p^2$

可解参数形式为

$$\begin{cases} t = cp^{\frac{a}{1-a}} + \frac{2}{1-2a}p \\ x = cap^{\frac{a}{1-a}} + \frac{1}{1-2a}p^2 \end{cases}$$ 加上特解 $x \equiv 0$.

若 $a=1$，则有 $\frac{dp}{dt}(2p+t) = 0$.

由 $\frac{dp}{dt} = 0$ 得 $p \equiv C \Rightarrow x = Ct + C^2$.

若 $2p+t=0$. 则 $p = -\frac{t}{2}$. $x = -\frac{t^2}{4}$.

例7：设 $a(t)$. $b(t) \in C(\alpha, \beta)$ 且

$|g(x) - g(\tilde{x})| \leq L|x - \tilde{x}|$ （$L > 0$）.

直接证明 (E)：$\begin{cases} \frac{dx}{dt} = a(t)g(x) + b(t) \\ x(t_0) = x_0 \end{cases}$

的解 $x(t; t_0, x_0)$ 在 (α, β) 上存在唯一.

证明：利用逐次逼近法.

$f(t,x)=a(t)g(x)+b(t)$ 关于 x 满足 Lipschitz 条件.

故满足局部存在唯一性. 下证整体性质

step 1：等价的积分方程问题

$$x(t)=x_0+\int_{t_0}^{t}[a(s)g(x(s))+b(s)]\,ds.$$

step 2：$\varphi_0(t)=x_0$.

$$\varphi_1(t)=x_0+\int_{t_0}^{t}[a(s)g(x_0)+b(s)]\,ds$$

$$\varphi_2(t)=x_0+\int_{t_0}^{t}[a(s)g(\varphi_1(s))+b(s)]\,ds$$

......

$$\varphi_n(t)=x_0+\int_{t_0}^{t}[a(s)g(\varphi_{n-1}(s))+b(s)]\,ds$$

易见 $\varphi_n(t)\in C(\alpha,\beta)\quad \forall n\in N$.

step 3：记 $M_1\triangleq\max\limits_{t\in[\alpha_1,\beta_1]}|a(t)|\quad M_2\triangleq\max\limits_{t\in[\alpha_1,\beta_1]}|b(t)|$

$N\triangleq(M_1|g(x_0)|+M_2)(\beta-\alpha)$

其中 $[\alpha_1,\beta_1]\subset(\alpha,\beta)$ 且 $t_0\in[\alpha_1,\beta_1]$.

当 $t\geq t_0$ 时. （$t\leq t_0$ 类似）

$$|\varphi_1(t)-\varphi_0(t)|\leq\int_{t_0}^{t}[|a(s)||g(x_0)|+|b(s)|]\,ds\leq N$$

$$|\varphi_2(t)-\varphi_1(t)|\leq\int_{t_0}^{t}|a(s)|\,|g(\varphi_1(s))-g(\varphi_0(s))|\,ds$$

$$\leq M_1NL(t-t_0)$$

下证 $\forall n\in N$. 有 $|\varphi_{n+1}(t)-\varphi_n(t)|\leq\frac{N}{n!}(M_1\cdot L)^n(t-t_0)^n$.

设 $n=k-1$ 时上式成立.

即 $|\varphi_k(t)-\varphi_{k-1}(t)| \le \frac{N}{(k-1)!}(M_1\cdot L)^{k-1}|t-t_0|^{k-1}$.

则 $|\varphi_{k+1}(t)-\varphi_k(t)| \le \int_{t_0}^{t}|a(s)||g(\varphi_k(s))-g(\varphi_{k-1}(s))|ds$

$$\le N M_1 L \int_{t_0}^{t}\frac{(M_1L)^{k-1}(s-t_0)^{k-1}}{(k-1)!}ds$$

$= N\frac{(M_1L)^k}{k!}(t-t_0)^k$. 从而 $n=k$ 时也成立.

故 $|\varphi_0(t)|+\sum_{k=1}^{\infty}|\varphi_k(t)-\varphi_{k-1}(t)|$

$$<|x_0|+\sum_{n=0}^{\infty}\frac{N}{n!}(M_1L)^n(\beta-\alpha)^n$$

$$=|x_0|+Ne^{M_1L(\beta-\alpha)}<+\infty.$$

从而 $\{\varphi_n(t)\}$ 内闭一致收敛. 记 $\varphi_n(t)\rightrightarrows\varphi(t)$.

Step 4: $\varphi_n(t)=x_0+\int_{t_0}^{t}[a(s)g(\varphi_{n-1}(s))+b(s)]ds$

两边 $n\to+\infty$. 则当 $t\in[\alpha,\beta]$ 时有

$$\varphi(t)=x_0+\int_{t_0}^{t}(a(s)g(\varphi(s))+b(s))ds.$$

从而原问题解在 (α,β) 上存在

Step 5: 再证唯一性

设 $\varphi(t)$ 与 $\varphi^*(t)$ 均为原初值问题的解

定义 $y(t)\triangleq|\varphi(t)-\varphi^*(t)|$. 则 $y(t_0)=0$.

$t\ge t_0$ 时 ($t\le t_0$ 类似)

$$y(t)\le\int_{t_0}^{t}|a(s)||g(\varphi(s))-g(\varphi^*(s))|ds$$

$$\le M_1L\int_{t_0}^{t}y(s)ds$$

记 $u(t)=\int_{t_0}^{t} y(s)\,ds$ 则有 $u'(t)=y(t)$.

$u(t_0)=0$, $u(t)\geq 0$ $(t\geq t_0)$.

则有 $u'(t)=y(t)\leq M_1 L\, u(t)$.

故 $\frac{d}{dt}\left[u(t)e^{-M_1 L(t-t_0)}\right]\leq 0$.

即 $u(t)e^{-M_1 L(t-t_0)} \leq u(t_0)=0$.

从而 $u(t)\leq 0$. 故有 $u(t)\equiv 0$.

即 $y(t)\equiv 0$ $(t\geq t_0)$.

$\varphi(t)\equiv\varphi^*(t)$ $(t\geq t_0)$. $t\leq t_0$ 时同理可证.

从而 $\varphi(t)\equiv\varphi^*(t)$ $\forall t\in(\alpha,\beta)$.

这就证明了解的唯一性.

第三次习题课

例1. $x = e^{y''} + y''$.

解：令 $u = y''$. 则 $x = e^u + u \Rightarrow dx = (e^u + 1)\,du$

$$y' = \int y''\,dx = \int u(e^u+1)\,du = (u-1)e^u + \frac{u^2}{2} + c_1$$

$$y = \int y'\,dx = \int \left((u-1)e^u + \frac{u^2}{2} + c_1\right)(e^u+1)\,du$$

$$= \left(\frac{u}{2} - \frac{3}{4}\right)e^{2u} + \left(\frac{u^2}{2} + c_1 - 1\right)e^u + \frac{u^3}{6} + c_1 u + c_2$$

例2：
$$\begin{cases} x'(t) = 1 - \dfrac{1}{y(t)} \\ y'(t) = \dfrac{1}{x(t) - t} \end{cases}$$

解：$y''(t) = \dfrac{1 - x'(t)}{(x(t)-t)^2} = \dfrac{1}{y}\cdot(y')^2$

$\Rightarrow y''y = (y')^2 \Rightarrow \dfrac{dy'}{dy}\cdot y'\cdot y = (y')^2$. $\because y' \neq 0$

$\therefore \dfrac{dy'}{y'} = \dfrac{dy}{y} \Rightarrow y' = c_1 y \Rightarrow \dfrac{dy}{y} = c_1\,dt$

$\Rightarrow y = c_2 e^{c_1 t}$ 代入得 $x = t + \dfrac{1}{c_1 c_2} e^{-c_1 t}$

例3：$xyy'' + xy'^2 = 3yy'$.

解：令 $z = yy'$. 则有 $xz' = 3z \Rightarrow \dfrac{dz}{dx} = \dfrac{3z}{x}$

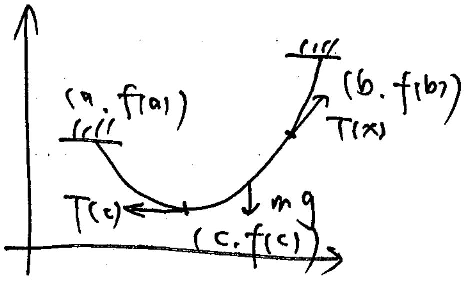
$\Rightarrow z = c_1 x^3 \Rightarrow yy' = c_1 x^3 \Rightarrow y^2 = \dfrac{c_1}{2}x^4 + c_2$.

其中 c_1, c_2 为常数. 特解包含在通解中

变分法简介

① 悬链线方程. 设绳线密度为 λ

记 $f(x)$ 为悬链线方程. 求 $f(x)$.



解法一: 对 $\widehat{cx}$ 段绳子受力分析.

$$\begin{cases} T(x)\cos\theta = T(c) \\ T(x)\,\mathrm{sh}\,\theta = mg = \lambda g \int_c^x \sqrt{1+|f'(t)|^2}\,dt \\ \tan\theta = f'(x) \end{cases}$$

$$\Rightarrow f'(x) = \frac{\lambda g}{T(c)} \int_c^x \sqrt{1+|f'(t)|^2}\,dt$$

$$\Rightarrow f''(x) = C\sqrt{1+|f'(x)|^2} \Rightarrow \frac{df'}{dx} = C\sqrt{1+(f')^2}$$

$$\Rightarrow \frac{df'}{\sqrt{1+f'^2}} = C\,dx \xrightarrow{f' \triangleq \mathrm{sh}\,\theta} \frac{\mathrm{ch}\,\theta\, d\theta}{\mathrm{ch}\,\theta} = C\,dx$$

$$\Rightarrow \theta = Cx + D \Rightarrow f'(x) = \mathrm{sh}(Cx+D)$$

$$\Rightarrow f(x) = \frac{1}{C}\mathrm{ch}(Cx+D) + D'.$$

解法二: $x \sim x+dx$ 段势能为 $\lambda g\sqrt{1+|f'(x)|^2}\, f(x)\, dx$

不妨设 $\lambda g = 1$. 则 $V(f) = \int_a^b \sqrt{1+|f'(x)|^2}\, f(x)\, dx$.

$f_t \triangleq f(x) + t g(x)$. $g \in C_0^\infty[a,b]$.

$$V(f_t) = \int_a^b \sqrt{1+|f'(x)+tg'(x)|^2}\,\big(f(x)+tg(x)\big)\,dx.$$

由能量最低原理知 $\left.\dfrac{d(V(f_t))}{dt}\right|_{t=0} = 0$.

从而 $\int_a^b \underbrace{\frac{f'(x)\,g'(x)\,f(x)}{\sqrt{1+|f'|^2}}}_{①} + \underbrace{g(x)\sqrt{1+|f'(x)|^2}}_{②}\,dx = 0$.

$$① = \int_a^b \frac{f'(x)\,f(x)\,d(g(x))}{\sqrt{1+|f'(x)|^2}} = \frac{f'(x)\,f(x)\,g(x)}{\sqrt{1+|f'(x)|^2}}\Bigg|_a^b - \int_a^b g(x)\,d\left(\frac{ff'}{\sqrt{1+|f'|^2}}\right)$$

$$= -\int_a^b g(x)\cdot\frac{\left[(f')^2+ff''\right]\sqrt{1+|f'|^2} - \frac{f'f''}{\sqrt{1+|f'|^2}}\cdot ff'}{1+|f'|^2}\,dx$$

$$= -\int_a^b g(x)\,\frac{(f')^2+ff''+(f')^4}{(1+|f'|^2)^{\frac{3}{2}}}\,dx$$

故有 $\int_a^b \left[\sqrt{1+|f'|^2} - \frac{(f')^2+ff''+(f')^4}{(1+|f'|^2)^{\frac{3}{2}}}\right] g(x)\,dx = 0$.

由 g 任意性知 $(1+|f'|^2)^2 = (f')^2 + ff'' + (f')^4$

$\Rightarrow 1+(f')^2 = ff'' \Rightarrow 1+(f')^2 = f\cdot\frac{df'}{df}\cdot f'$

$\Rightarrow \frac{1}{2}\ln(1+|f'|^2) = \ln f + C_1$

$\Rightarrow (f')^2+1 = c^2f^2$. 从而 $f' = \sqrt{c^2f^2-1}$.

$\Rightarrow \frac{df}{\sqrt{c^2f^2-1}} = dx \xRightarrow{f=\frac{\operatorname{ch}\theta}{c}} \frac{\frac{\operatorname{sh}\theta}{c}\,d\theta}{\operatorname{sh}\theta} = dx$

$\Rightarrow \theta = cx+D \Rightarrow f(x) = \frac{1}{c}\operatorname{ch}(cx+D)$.

② Euler - Lagrange 方程

设 $L(q, q', t)$ 为光滑函数 $\gamma: [a,b] \to U$.

定义 $S(\gamma) = \int_a^b L(\gamma(t), \gamma'(t), t)\,dt$.

γ 为极值点时有 $\frac{d}{dt}\frac{\partial L}{\partial q'} = \frac{\partial L}{\partial q}$.

证明：$\phi_s(t) \triangleq \gamma(t) + s\,v(t)$. $v \in C_0^\infty[a,b]$

$$\frac{dS(\phi_s)}{ds} = \frac{d}{ds}\int_a^b L(\phi_s(t), \phi_s'(t), t)\,dt$$

$$= \int_a^b \underset{①}{\frac{\partial L}{\partial q}\cdot v(t)} + \underset{②}{\frac{\partial L}{\partial q'}\cdot v'(t)}\,dt$$

$$② = \frac{\partial L}{\partial q'}v(t)\Big|_a^b - \int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial q'}\right)v(t)\,dt$$

$$= -\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial q'}\right)v(t)\,dt$$

故 $\int_a^b \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial q'}\right)\right)v(t)\,dt$

由 v 任意性知 $\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial q'}\right)$.

③ Poisson 方程. Ω 有界区域

求 $\inf\limits_{u\in A}\int_\Omega |\nabla u|^2 + 2uf\,dx$ 取最小值时的 u.

其中 $A = \{w \mid w = \varphi \text{ on } \partial\Omega\}$.

$$I[u] \triangleq \int_\Omega |\nabla u|^2 + 2uf\,dx. \quad v \in C_0^\infty(\Omega)$$

$$I[u+tv] = \int_\Omega |\nabla u + t\nabla v|^2 + 2(u+tv)f\,dx$$

$$= \int_\Omega |\nabla u|^2 + 2t\nabla u\cdot\nabla v + t^2|\nabla v|^2 + 2uf + 2tvf\,dx$$

$$\frac{dI}{dt}\Big|_{t=0} = \int_\Omega \underset{①}{2\nabla u\cdot\nabla v} + \underset{②}{2vf}\,dx$$

$$① = 2\int_{\partial\Omega}\frac{\partial u}{\partial n}\cdot v\,dS - 2\int_\Omega \Delta u\cdot v\,dx = -2\int_\Omega \Delta u\cdot v\,dx$$

代入得 $\int_\Omega (\Delta u - f)\, dx = 0$. 由 v 任意性知

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

以下证明 $\forall w \in A$. 有 $I[w] \geq I[u]$.

事实上 $I[w] = \int_\Omega |\nabla w|^2 + 2wf\, dx$

$$= \int_\Omega |\nabla(w-u) + \nabla u|^2 + 2uf + 2(w-u)f\, dx$$

$$= \int_\Omega |\nabla(w-u)|^2 + |\nabla u|^2 + 2\nabla(w-u)\cdot\nabla u + 2uf + 2(w-u)f\, dx$$

$$\geq I[u] + \int_\Omega 2\nabla(w-u)\cdot\nabla u + 2(w-u)f\, dx$$

$$= I[u] + 2\int_{\partial\Omega} (w-u)\frac{\partial u}{\partial n}\, ds = I[u].$$

④ 测地线第一变分.

直线: $\gamma(t) = at+b$. $\gamma'(t) = a$. $\gamma''(t) = 0$.

定义: 若 $\gamma(t)$ 满足 $D_{\frac{\partial}{\partial t}}\gamma'(t) = (\gamma''(t))^T = 0$.

则称 $\gamma(t)$ 为测地线.

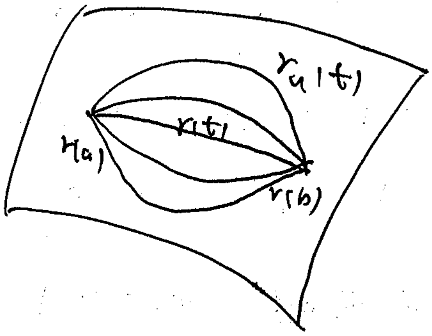
命题: $|\gamma'(t)| \equiv \text{const}$.

证明: $\frac{d}{dt}\langle \gamma'(t), \gamma'(t)\rangle = 2\langle D_{\frac{\partial}{\partial t}}\gamma'(t), \gamma'(t)\rangle = 0$.

从而 $\langle \gamma'(t), \gamma'(t)\rangle \equiv \text{const}$. 即 $|\gamma'(t)|^2 \equiv \text{const}$.

从而得证.

设 $r_u(t)$ 连接 $r(a)$, $r(b)$



定义 $U(t)=\frac{\partial}{\partial u} r_u(t)\Big|_{u=0}$

则 $U(a)=\frac{\partial}{\partial u} r_u(a)\Big|_{u=0}=0$

$U(b)=\frac{\partial}{\partial u} r_u(b)\Big|_{u=0}=0$.

定理：$r_u(t)$ 中连接 $r(a)$, $r(b)$ 的最短线 $r(t)$ 为测地线.

证明：$L(u)=\int_a^b |r_u'(t)|\,dt=\int_a^b \langle r_u'(t), r_u'(t)\rangle^{\frac{1}{2}}\,dt$

$$L'(u)=\int_a^b \frac{d}{du}\langle r_u'(t), r_u'(t)\rangle^{\frac{1}{2}}\,dt$$

$$=\int_a^b \frac{2\langle \frac{\partial}{\partial u}\frac{\partial}{\partial t} r_u(t), r_u'(t)\rangle}{2|r_u'(t)|}\,dt=\int_a^b \frac{\langle \frac{\partial}{\partial t}\frac{\partial}{\partial u} r_u(t), r_u'(t)\rangle}{|r_u'(t)|}\,dt$$

$$0=L'(0)=\int_a^b \frac{\langle \frac{\partial}{\partial t}U(t), r'(t)\rangle}{|r'(t)|}\,dt$$

$$=\frac{1}{|r'|}\left(\int_a^b \frac{\partial}{\partial t}\langle U(t), r'(t)\rangle\,dt-\int_a^b \langle U(t), D_{\frac{\partial}{\partial t}} r'(t)\rangle\,dt\right)$$

$$=\frac{1}{|r'|}\left(\langle U(t), r'(t)\rangle\Big|_a^b-\int_a^b \langle U(t), D_{\frac{\partial}{\partial t}} r'(t)\rangle\,dt\right)$$

$$=-\frac{1}{|r'|}\int_a^b \langle U(t), D_{\frac{\partial}{\partial t}} r'(t)\rangle\,dt$$

由 $U(t)$ 任意性知 $D_{\frac{\partial}{\partial t}} r'(t)=0$.

即 $r(t)$ 为测地线.

第四次 习题 课

①. 比较定理的应用

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \mu(x^2-1)y \end{cases} \quad \mu > 0$$

试证它的积分曲线从 y 负半轴任一点A出发左行时必与 x 轴负半轴相交

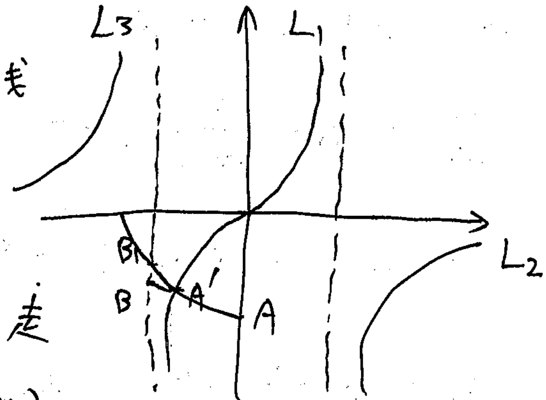
证明: $\dfrac{dy}{dx} = \dfrac{-x + \mu(x^2-1)y}{y} \triangleq f(x,y)$

则 $f(x,y)$ 在 $G = \{(x,y) \mid -\infty < x < +\infty, y<0\}$ 连续

满足局部 Lipschitz 条件

水平等倾线 $L: Q(x,y) \triangleq -x - \mu(x^2-1)y = 0$

三支 L_1, L_2, L_3 如图所示



$\forall y_0 < 0$, 过 $A(0, y_0)$ 的左行解曲线

(1) $y = y(x)$ 与 L_1 相交

在 L_1 与 L_2 之间 $y<0$, 有 $\dfrac{dy}{dx} > 0$

故由 A 出发的解曲线 Γ 往左下方向走

取 $\tilde{f}(x,y) \triangleq -\mu(x^2-1) \geq f(x,y)$

$$\begin{cases} \dfrac{dy}{dx} = -\mu(x^2-1) \\ y(0) = y_0 \end{cases} \Rightarrow y(x) = \mu x - \frac{1}{3}\mu x^3 + y_0$$

与 $x=-1$ 交 $B(-1, y_1)$ $\quad y_1 = -\frac{2}{3}\mu - y_0$

从而与 L_1 相交, 由第二比较定理知与 L' 相交于 A'

(2) 过 A' 的 Γ 或与 $x=-1$ 在 $-1<x<0$ 内与 x 轴负半轴相交

L_1 与 L_3 之间 $y<0$, 有 $\dfrac{dy}{dx} < 0$ 向左上方走

1° 与 x 负半轴 在 $-1<x<0$ 内交于 P点

2° 与 $x=-1$ 交于 B_1 (位于B上方)

(3) 过 B_1 的 Γ' 与 x 负半轴相交

$\tilde{f}(x,y) \triangleq -\dfrac{x}{y} \geq f(x,y)$

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(-1) = y_2 \end{cases} \Rightarrow y_2^2 + 1 = x^2 + y^2 \Rightarrow x \text{ 交于 } C(-\sqrt{1+y_2^2}, 0)$$

$\Rightarrow \Gamma$ 交 x 于 C 右方 C_1.

②. 设 $x = \varphi(t; x_0, y_0)$ $y = \psi(t; x_0, y_0)$

$$\begin{cases} \dfrac{dx}{dt} = xy + t^2 \\ 2\dfrac{dy}{dt} = -y^2 \end{cases}$$ 适合 $\varphi(1; x_0, y_0) = x_0$, $\psi(1; x_0, y_0) = y_0$.

讨论 解对初值的连续依赖性

(1) $y \equiv 0$ 特解 $x = \dfrac{t^3}{3} + C$

$\psi(t; x_0, 0) \equiv 0$, $\varphi(t; x_0, 0) = \dfrac{t^3}{3} + x_0 - \dfrac{1}{3}$

φ、ψ 均为 (x_0, y_0) 的连续函数.

(2) $y \neq 0$, 则有 $y = \dfrac{2}{t+c_1}$, $\Rightarrow y_0 = \dfrac{2}{1+c_1}$

$$\frac{dx}{dt} = x \cdot \frac{2}{t+c_1} + t^2$$

$$\Rightarrow x = (t+c_1)^2 \left[c_2 + t - \frac{c_1^2}{t+c_1} - 2c_1 \ln(t+c_1) \right]$$

而 $c_1 = \dfrac{2}{y_0} - 1$. 又由

$$x_0 = (1+c_1)^2 \left(c_2 + 1 - \frac{c_1^2}{1+c_1} - 2c_1 \ln(1+c_1) \right)$$

有 $c_2 = \dfrac{x_0 y_0^2}{4} - 1 + \dfrac{y_0\left(\frac{2}{y_0} - 1\right)^2}{2} + 2\left(\frac{2}{y_0} - 1\right) \ln\left(\frac{2}{y_0}\right)$

代入得 x、y 的表达式. 易见其关于 (x_0, y_0) 连续.

例1: $$\frac{dx}{e^x + z} = \frac{dy}{e^y + z} = \frac{dz}{z^2 - e^{x+y}}$$

解: 由 $$\frac{e^y\, dx}{e^y(e^x + z)} = \frac{dz}{z^2 - e^{x+y}} = \frac{dy}{e^y + z}$$

$$\Rightarrow \frac{e^y dx + dz}{e^{x+y} + ze^y + z^2 - e^{x+y}} = \frac{dy}{e^y + z}$$

$$\Rightarrow \frac{e^y dx + dz}{ze^y + z^2} = \frac{dy}{e^y + z} \Rightarrow \frac{e^y dx + dz}{z} = dy$$

$$\Rightarrow dx - ze^{-y} dy + e^{-y} dz = 0$$

$$\Rightarrow x + ze^{-y} = c_1 .$$

类似有 $y + ze^{-x} = c_2$. 从而得到通积分

例 2：
$$\begin{cases} y' = \dfrac{z}{x} & \cdots ① \\ z' = \dfrac{(y-z)^2 + xz}{x^2} & \cdots ② \end{cases}$$

解： $z = xy'$ 代入 ② 中有

$$y' + xy'' = \frac{(y - xy')^2 + x^2 y'}{x^2} \Rightarrow y'' = \frac{1}{x^3}(y - xy')^2$$

$$\Rightarrow x^3 d^2 y = (y dx - x dy)^2$$

令 $x = e^t$. $y = ue^t$. 则有

$$dx = e^t dt \quad dy = du\, e^t + ue^t dt \Rightarrow y' = u' + u$$

$$y'' = e^{-t}(u'' + u')$$

从而有 $u'' + u' - (u')^2 = 0$. 令 $u' = p$.

则 $p' = p^2 - p \Rightarrow p = \dfrac{1}{1 - c_1 e^t}$

即 $u = t + c_2 - \ln(1 - c_1 e^t)$

$$y = (c_2 + \ln x)x - x\ln(1 - c_1 x)$$

又得 $z = xy' = (c_2 + 1)x + x\ln\dfrac{x}{1 - c_1 x} + \dfrac{c_1 x^2}{1 - c_1 x}$

从而得到原方程的通解

例3：
$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{2xy}{y^2+z^2-x^2} \\ \dfrac{dz}{dx} = -\dfrac{2xz}{y^2+z^2-x^2} \end{cases}$$

解：
$$\frac{dx}{-(y^2+z^2-x^2)} = \frac{dy}{2xy} \overset{(*)}{=} \frac{dz}{2xz}$$

由(*)知 $\frac{y}{z} = c_1$. 作组合

$$\frac{x\,dx + y\,dy + z\,dz}{-x(y^2+z^2-x^2) + 2xy^2 + 2xz^2} = \frac{dy}{2xy}$$

$$\Rightarrow \frac{\frac{1}{2}d(x^2+y^2+z^2)}{x(x^2+y^2+z^2)} = \frac{dy}{2xy}$$

$$\Rightarrow \frac{x^2+y^2+z^2}{y} = c_2$$

从而通积分为
$$\begin{cases} \dfrac{y}{z} = c_1 \\ \dfrac{x^2+y^2+z^2}{y} = c_2 \end{cases}$$

例4：
$$\begin{cases} \dfrac{dx}{dt} = x + x(x+y) \\ \dfrac{dy}{dt} = z + y(x+y) \\ \dfrac{dz}{dt} = y + z(x+y) \end{cases}$$

解：由 $\frac{dx}{dt} = x(x+y+1)$ $\quad \frac{d(y+z)}{dt} = (y+z)(x+y+1)$

$$\Rightarrow \frac{dx}{x} = \frac{d(y+z)}{y+z}$$ 得首次积分 $y+z = c_1 x$.

又由 $\frac{d(y-z)}{dt} = (y-z)(x+y-1)$

$$\Rightarrow \frac{d(y-z)}{y-z} = (x+y-1)dt = (x+y+1)dt - 2dt$$

$$= \frac{dx}{x} - 2dt$$ 得 $y-z = c_2 x e^{-2t}$

由 $y+z$ 与 $y-z$ 的表达式有

$$\begin{cases} y=\frac{1}{2}(c_1+c_2e^{-2t})x & \cdots\cdots ① \\ z=\frac{1}{2}(c_1-c_2e^{-2t})x & \cdots ② \end{cases}$$

$$\frac{dx}{dt}=x+x^2\left(1+\frac{1}{2}c_1+\frac{1}{2}c_2e^{-2t}\right)$$

$$\Rightarrow -\frac{d}{dt}\left(\frac{1}{x}\right)=\frac{1}{x}+1+\frac{1}{2}c_1+\frac{1}{2}c_2e^{-2t}$$

$$\Rightarrow \frac{1}{x}=e^{-t}\left\{c_3-e^t-\frac{1}{2}c_1e^t+\frac{1}{2}c_2e^{-t}\right\}$$

代入有

$$y=\frac{(c_1+c_2e^{-2t})e^t}{c_3-e^t-\frac{1}{2}c_1e^t+\frac{1}{2}c_2e^{-t}}$$

$$z=\frac{c_1e^t-c_2e^{-t}}{c_3-e^t-\frac{1}{2}c_1e^t+\frac{1}{2}c_2e^{-t}}$$

其中 c_1, c_2, c_3 为常数.

第五次习题课

I. 极坐标下的 Laplace 方程 $\Delta u = 0$

分离变量 $u(r,\theta) = R(r)\Theta(\theta)$.

则 $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

$\Rightarrow \begin{cases} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{cases}$. 又易见 $u(r,\theta) = u(r,\theta+2\pi)$

① $\lambda = 0$. 则 $\Theta(\theta) = A + B\theta$. 故 $\Theta(\theta) = A$.

($\because \Theta(0) = \Theta(2\pi) \Rightarrow B = 0$)

记作 $\lambda_0 = 0$. 此时 $\Theta_0(\theta) = A_0$.

② $\lambda > 0$. 则有 $\Theta(\theta) = A\cos k\theta + B\sin k\theta$. ($k = \sqrt{\lambda}$)

由 $\Theta(0) = \Theta(2\pi) \Rightarrow k \in N^*$. 故 $\lambda_n = n^2$.

其中 $n \in N^*$. $\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$.

③ $\lambda < 0$. $\Theta(\theta) = Ae^{\sqrt{-\lambda}\theta} + Be^{-\sqrt{-\lambda}\theta}$.

若要 $\Theta(0) = \Theta(2\pi)$. 则 $A = B = 0$.

将 λ_n 代入方程 有

$r^2R_n'' + rR_n' - n^2R_n = 0$. $(n \in N)$. 为欧拉方程

求解: 由 $\rho^2 - n^2 = 0$. $(\alpha = 1, \beta = -n^2)$

当 $n = 0$ 时 $\rho = 0$. $\Rightarrow R_0(r) = c_0 + d_0\ln r$.

当 $n \in N^*$ 时 $\rho_1 = n$, $\rho_2 = -n$.

$R_n = c_n\left(\frac{r}{a}\right)^n + d_n\left(\frac{r}{a}\right)^{-n}$

而 $r \to 0$ 时 $u(0,\theta)$ 为有限数.

从而 $d_0 = 0$, $d_n = 0$.

$\Rightarrow R_0 = c_0 . \quad R_n(r) = c_n \frac{r^n}{a^n}$

得到本征解 $u_0(r,\theta) = a_0$.

$u_n(r,\theta) = \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta) \quad n \in N^*$.

一般解为 $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta)$

考虑边界条件 $u(a,\theta) = f(\theta)$

$\Rightarrow f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$

此即 f 的 Fourier 展开 .

Ⅱ. 波方程的分离变量法

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0<x<L,\ t>0 \\ u|_{t=0} = \varphi(x) \quad u_t|_{t=0} = \psi(x) & 0 \le x \le L \\ u_x|_{x=0} = 0 ,\ u|_{x=L} = 0 & t>0 \end{cases}$$.

解： $u(x,t) = X(x)\,T(t)$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \quad X(L) = 0 \\ T'' + \lambda a^2 T = 0 \end{cases}$$.

① $\lambda = 0$. $X(x) = Ax + B$. $B = X'(0) = 0$. $A = X(L) = 0$.

② $\lambda < 0$ $X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

$$\left.\begin{array}{l} X(L) = 0 \Rightarrow Ae^{\sqrt{-\lambda}L} + Be^{-\sqrt{-\lambda}L} = 0 \\ X'(0) = 0 \Rightarrow A - B = 0 \end{array}\right\} \Rightarrow \begin{cases} A = 0 \\ B = 0 \end{cases}$$.

从而 $\lambda \le 0$ 时原方程只有零解

③ $\lambda > 0$. $X(x) = A\cos kx + B\sin kx \quad k=\sqrt{\lambda}$

A、B 为常数 $X'(x) = -kA\sin kx + kB\cos kx$

由 $X'(0) = 0 \Rightarrow B = 0$. $X(x) = A\cos kx$.

$X(L) = 0 \Rightarrow kL = \frac{2n+1}{2}\pi$, $n \in N$.

$$\lambda_n = \left[\frac{(2n+1)\pi}{2L}\right]^2 \qquad X_n(x) = A_n \cos\frac{(2n+1)\pi}{2L}x$$

将 λ_n 代入 (2-3) 有 $T_n(t) = c_n\cos\frac{(2n+1)\pi a}{2L}t + d_n\sin\frac{(2n+1)\pi a}{2L}t$

故 $u(x,t) = \sum_{n=0}^{\infty}\left[C_n\cos\frac{(2n+1)\pi a}{2L}t + D_n\sin\frac{(2n+1)\pi a}{2L}t\right]\cos\frac{(2n+1)\pi}{2L}x$

其中 $C_n = \frac{2}{L}\int_0^L \phi(x)\cos\frac{(2n+1)\pi}{2L}x\,dx$

$$D_n = \frac{4}{(2n+1)\pi a}\int_0^L \psi(x)\cos\frac{(2n+1)\pi}{2L}x\,dx$$

Ⅲ. 热方程的分离变量法

$$\begin{cases} \Delta u = 0 & (0<x<a \quad 0<y<b) \\ u(x,0) = 0,\ u(x,b) = T & (0 \le x \le a) \\ u(0,y) = 0,\ u(a,y) = 0 & (0 \le y \le b) \end{cases}$$

解：令 $u(x,y) = X(x)Y(y)$

则 $\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$

且 $X(x)Y(0) = 0 \qquad X(x)Y(b) = T$

$X(0)Y(y) = 0 \qquad X(a)Y(y) = 0$

$\because Y \neq 0$. 故 $X(0)=X(a)=0$.

考虑 $\begin{cases} X''+\lambda X=0 \\ X(0)=0,\ X(a)=0 \end{cases}$ （$\lambda \le 0$ 时只有零解）

$\Rightarrow \lambda_n = \left(\frac{n\pi}{a}\right)^2 \quad X_n = b_n \sin\frac{n\pi}{a}x,\quad n\in N^*$.

将 λ_n 代入 $Y''-\lambda Y=0$ 有

$Y_n(y) = c_n \,\mathrm{ch}\frac{n\pi}{a}y + d_n\,\mathrm{sh}\frac{n\pi}{a}y$

由 $Y_n(0)=0 \Rightarrow c_n=0$.

故 $Y_n(y) = d_n\,\mathrm{sh}\frac{n\pi}{a}y$.

从而 $u_n(x,t) = X_n(x)Y_n(y) = B_n \sin\frac{n\pi}{a}x\,\mathrm{sh}\frac{n\pi}{a}y$

$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x\,\mathrm{sh}\frac{n\pi}{a}y$.

则 $T = u(x,b) = \sum_{n=1}^{\infty} B_n\,\mathrm{sh}\frac{n\pi b}{a}\sin\frac{n\pi}{a}x$

由 $B_n\,\mathrm{sh}\frac{n\pi b}{a} = \frac{2}{a}\int_0^a T\sin\frac{n\pi x}{a}dx$

$= \frac{2T}{n\pi}\int_0^{n\pi}\sin y\,dy = \begin{cases} 0 & 2\mid n \\ \frac{4T}{n\pi} & 2\nmid n \end{cases}$.

则有 $B_n = \frac{4T}{n\pi}\,\mathrm{ch}\frac{n\pi b}{a} \qquad 2\nmid n$.

例如 $a=b=1$ 时可计算得

$u\left(\frac{1}{2},\frac{1}{2}\right) = \frac{4T}{\pi}\sum_{k=0}^{\infty}\frac{\mathrm{sh}\frac{(2k+1)\pi}{2}\sin\frac{(2k+1)\pi}{2}}{(2k+1)\,\mathrm{sh}(2k+1)\pi}$

$= \frac{2T}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)\,\mathrm{ch}\frac{(2k+1)\pi}{2}}$

例 1： $y^{(4)}+2y''+y=\sin x$

$y(0)=1,\ y'(0)=-2,\ y''(0)=3,\ y'''(0)=0$

解： $\lambda^4+2\lambda^2+1=0 \Rightarrow \lambda=\pm i$.

基解为 $\cos x,\ \sin x,\ x\cos x,\ x\sin x$.

$\tilde{y}=x^2(a\cos x+b\sin x)$ 代入计算得

$\tilde{y}^{(4)}+2\tilde{y}''+\tilde{y}=\sin x \Rightarrow a=0,\ b=-\frac{1}{8}$

$\Rightarrow \tilde{y}=-\frac{1}{8}x^2\sin x$ 为特解. 代入初值得解为

$$y=\left(1+\frac{5}{8}x\right)\cos x+\left(-\frac{21}{8}+2x-\frac{1}{8}x^2\right)\sin x$$

例 2： $y'''+y''+y'+y=e^x+e^{-x}+\sin x$

解： $\lambda^3+\lambda^2+\lambda+1=0$

$\Rightarrow \lambda_1=1,\ \lambda_2=i,\ \lambda_3=-i$.

基本解为 $e^x,\ \sin x,\ \cos x$

易见对 $e^x,\ e^{-x},\ \sin x$ 的特解分别可取为

$\frac{1}{4}e^x,\ \frac{x}{2}e^{-x},\ -\frac{1}{4}x(\cos x+\sin x)$

从而 $y=c_1e^x+c_2\cos x+c_3\sin x+\frac{1}{4}e^x+\frac{x}{2}e^{-x}-\frac{1}{4}x(\cos x+\sin x)$

例 3： $\begin{cases} x''+y'+x=e^t & \cdots ① \\ y''+x'+y=0 & \cdots ② \end{cases}$

解： ①+② 得

$$\frac{d^2(x+y)}{dt^2}+\frac{d(x+y)}{dt}+(x+y)=e^t$$

令 $u=x+y$ 则 $u''+u'+u=e^t$

可解出 $u=\frac{1}{3}e^t+e^{-\frac{t}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}t+c_2\sin\frac{\sqrt{3}}{2}t\right)$

①－② 得 $v''-v'+v=e^t$. 其中 $v=x-y$.

可解出 $v=e^{\frac{t}{2}}\left(c_3\cos\frac{\sqrt{3}}{2}t+c_4\sin\frac{\sqrt{3}}{2}t\right)+e^t$.

从而由 $\begin{cases} x=\frac{u+v}{2} \\ y=\frac{u-v}{2} \end{cases}$ 可得 x, y 的表达式.

例4: $A=\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ $\vec{f}(x)=\begin{pmatrix} x^2 \\ 2x \\ x \end{pmatrix}$

解: 易见 $\lambda_1=\lambda_2=\lambda_3=-1$

$A+I=\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $(A+I)^2=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $(A+I)^3=0$.

易知基解矩阵为 $\Phi(x)=\begin{pmatrix} 1 & -x & x^2 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}e^{-x}$

从而 $\Phi^{-1}(x)=\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}e^{x}$

$$\int\Phi^{-1}(x)f(x)\,dx=\int\begin{pmatrix} 3x^2 \\ 2x+x^2 \\ x \end{pmatrix}e^x\,dx=\begin{pmatrix} 3x^2-6x+6 \\ x^2 \\ x-1 \end{pmatrix}e^x$$

$$\Phi(x)\int\Phi^{-1}(x)f(x)\,dx=\begin{pmatrix} 2x^2-6x+6 \\ x \\ x-1 \end{pmatrix}$$

故有 $y=C\,\Phi(x)+\begin{pmatrix} 2x^2-6x+6 \\ x \\ x-1 \end{pmatrix}$

$$=C\begin{pmatrix} 1 & -x & x^2 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}e^{-x}+\begin{pmatrix} 2x^2-6x+6 \\ x \\ x-1 \end{pmatrix}$$

第六次习题课

1. Lyapunov 稳定性

例：讨论 $x''+px+q(x-x^3)=0$ 的零解

$x=0$，$x'=0$ 的稳定性

解：方程化为
$$\begin{cases}\dfrac{dx}{dt}=y\\[2mm] \dfrac{dy}{dt}=-qx-py+qx^3\end{cases}$$

用 $V(x,y)=\frac{q}{2}x^2+\frac{1}{2}y^2-\frac{q}{4}x^4$ 来判定

在 $(0,0)$ 附近 $V>0$.

$$\frac{dV}{dt}=y\frac{\partial V}{\partial x}+(-qx-py+qx^3)\frac{\partial V}{\partial y}$$
$$=y(qx-qx^3)+y(-qx-py+qx^3)=-py^2$$

$$\begin{cases}p>0 & \dfrac{dV}{dt}<0. & \text{零解渐近稳定}\\[2mm] p=0 & \dfrac{dV}{dt}=0 & \text{零解稳定但不渐近稳定}\\[2mm] p<0 & \dfrac{dV}{dt}>0 & \text{零解不稳定}\end{cases}$$

2.

$$\begin{cases}\Delta u=2 & \text{in } \Omega\subset R^n\\ u=0 & \text{on } \partial\Omega\end{cases}$$

解：令 $\varphi=|\nabla u|^2-2\alpha u$

则有 $\varphi_i=2u_lu_{li}-2\alpha u_i$

$$\Delta\varphi=2u_{li}^2+u_l(\Delta u)_l-2\alpha\Delta u$$
$$=2|D^2u|^2-4\alpha$$

$\geq 2(u_{11}^2 + u_{22}^2) - 4\alpha$

$\geq (u_{11}+u_{22})^2 - 4\alpha = 4 - 4\alpha \geqslant 0 \quad (\alpha \leq 1$时$)$

从而 $\max\limits_{\Omega} \varphi \leq \max\limits_{\partial\Omega} \varphi$.

命题：$|\nabla u|^2 |D^2 u|^2 = |\nabla u|^2 (\Delta u)^2 + \frac{1}{2} \sum\limits_{i=1}^{2} \left(|\nabla u|^2_{\ i} \right)^2$

$- \Delta u \sum u_i |\nabla u|^2_{\ i}$.

证明：只要证

$|\nabla u|^2 |D^2 u|^2 = |\nabla u|^2 (\Delta u)^2 + \frac{1}{2} (2u_l u_{li})^2$

$- \Delta u \; u_i (2u_l u_{li}) = \underset{①}{|\nabla u|^2 (\Delta u)^2}$

$+ \underset{②}{2(u_l u_{li})^2} - \underset{③}{2\Delta u \; u_i u_l u_{li}}$. 即可

$② + ③ = 2(u_1 u_{11} + u_2 u_{21})^2 + 2(u_1 u_{12} + u_2 u_{22})^2$

$-2(u_{11}+u_{22})(u_1^2 u_{11} + u_2^2 u_{22} + 2u_1 u_2 u_{12})$

$= 2u_1^2 u_{12}^2 + 2u_2^2 u_{12}^2 - 2u_1^2 u_{11} u_{22} - 2u_2^2 u_{11} u_{22}$

$① + ② + ③ = (u_1^2 + u_2^2)(u_{11}+u_{22})^2 + 2u_1^2 u_{12}^2$

$+ 2u_2^2 u_{12}^2 - 2(u_1^2 + u_2^2) u_{11} u_{22}$

$= (u_1^2 + u_2^2)(u_{11}^2 + u_{22}^2 + 2u_{12}^2) = |\nabla u|^2 |D^2 u|^2$

3. 求下面问题解参数形式

$$\begin{cases} -y u_x + x u_y = 0 \\ u(s, s^2) = s^3 \quad (s > 0) \end{cases} \qquad ①$$

解：由
$$\begin{cases} \frac{d}{dt} X(s,t) = -Y(s,t) \\ \frac{d}{dt} Y(s,t) = X(s,t) \end{cases} \qquad ②$$

由初始条件 $X(s,0)=s$，$Y(s,0)=s^2$.

可得特征线，$(X(s,t), Y(s,t))$

② 的通解 $\begin{cases} X(s,t) = c_1(s)\cos t + c_2(s)\sin t \\ Y(s,t) = c_1(s)\sin t - c_2(s)\cos t \end{cases}$

由初始条件 $c_1(s) = s$，$c_2(s) = -s^2$.

对①的偏微有 $c(x,y)=0$ 和 $f(x,y)=0$

因此 $c(s,t) = c(X(s,t), Y(s,t))$

$m(s,t) = e^{\int_0^t c(s,\tau)\,d\tau} = 0$

$F(s,t) = f(X(s,t), Y(s,t)) = 0$.

$G(s) = s^3$. $u(s,t) = \frac{1}{m}\left(\int_0^t mF\,d\tau + G(s)\right) = s^3$.

$X(s,t) = s\cos t - s^2\sin t$

$Y(s,t) = s\sin t + s^2\cos t$

$u(s,t) = s^3$.

4．求解 $\begin{cases} 2u_x + 3u_y + 5u_z - u = 0 \\ u(x,y,0) = x^2\sin y \end{cases}$

$\frac{dy}{dx} = \frac{3}{2}$. $\frac{dz}{dx} = \frac{5}{2}$

得 $y = \frac{3}{2}x + \frac{\alpha}{2}$，$z = \frac{5}{2}x + \frac{\beta}{2}$ α, β 为任意常数

令 $\begin{cases} \bar{x} = 2y - 3x \\ \bar{y} = 2z - 5x \\ \bar{z} = z \end{cases}$ 显然 Jacobi 不为 0

令 $\bar{u}(\bar{x},\bar{y},\bar{z}) = u(x,y,z)$ 得

$2u_x+3u_y+5u_z = 5\bar{u}_{\bar{z}} \Rightarrow 5\bar{u}_{\bar{z}} = \bar{u}$.

从而 $\bar{u}(\bar{x},\bar{y},\bar{z}) = c(\bar{x},\bar{y})e^{\frac{\bar{z}}{5}}$

其中 $c(\bar{x},\bar{y})$ 为 $(\bar{x},\bar{y})$ 的任意 c^1 函数

则 $u(x,y,z) = C(2y-3x, 2z-5x)e^{\frac{z}{5}}$

由于 $u(x,y,0) = x^2\sin y$

$\Rightarrow C(2y-3x, -5x) = x^2\sin y$

$\Rightarrow C(r,s) = (-\frac{s}{5})^2 \sin(\frac{r}{2}-\frac{3}{10}s)$

$u(x,y,z) = (x-\frac{2}{5}z)^2\sin(y-\frac{3}{5}z)e^{\frac{z}{5}}$

求解爆破时间

回顾: Cauchy 问题 $\begin{cases} u_t + a(u)u_x = 0 \\ u|_{t=0} = \varphi(x) \end{cases}$ ①

$a(u)\in C^1$ $\quad \varphi\in C^1$ 且 $\|\varphi\|_{C^1} < +\infty$.

定义. 上面问题解 $u=u(t,x)$

若 $|u_x|$ 在有限时间内趋于无穷大. 则称为导数的突变. 突变形成的最早时刻 $t_0\ge 0$ 称为爆破时间

由局部解存在定理. ① 存在局部解 (C^1解)

对解存在范围内任一给定点 (t,x). 通过该点可以向下作一条特征线. 记为 $x=\xi(\tau;t,x)$

注意到 a 的 C^1 光滑性和初值 φ 下 C^1 模有界

可知特征线斜率有界. 故与 x 轴相交

记其交点为 $(0,\alpha)$ 于是 $u(t,x)=\varphi(\alpha)$ ②

$\xi(\tau;t,x) = \alpha + a(\varphi(\alpha))\tau$. ③

当 $\tau=t$ 时 $\quad x = \alpha + a(\varphi(\alpha))t$

②式两端求导（关于 x） $u_x = \varphi'(\alpha)\frac{\partial \alpha}{\partial x}$

③式关于 α 求导有 $\frac{\partial x}{\partial \alpha} = 1 + \frac{da}{d\alpha}t$.

$$u_x = \frac{\varphi'(\alpha)}{1 + \frac{da(\varphi(\alpha))}{d\alpha}t}$$ 讨论何时成为无穷大.

转化为 $1 + \frac{da(\varphi(\alpha))}{d\alpha}t$ 何时为 0 的问题.

如果 $\frac{da(\varphi(\alpha))}{d\alpha} \geq 0$. $\forall \alpha \in \mathbb{R}$.

当 $t > 0$ 时. $1 + \frac{da}{d\alpha}t \geq 1$ 不会导数突变.

若 $\exists \alpha$ s.t. $\frac{da(\varphi(\alpha))}{d\alpha} < 0$. 则会突变

易见最早爆破时间所对应下的 α_0 应使

$\frac{da(\varphi(\alpha))}{d\alpha}$ 取最小值

$$t_b = -\frac{1}{\left.\frac{da(\varphi(\alpha))}{d\alpha}\right|_{\alpha=\alpha_0}}$$

例：求 $\begin{cases} u_t + uu_x = 0 \\ u|_{t=0} = \sin x \end{cases}$ ⓐ

$\begin{cases} u_t + u^2u_x = 0 \\ u|_{t=0} = \frac{1}{1+x^2} \end{cases}$ ⓑ

的爆破时间和地点.

解：ⓐ $a(u) = u$. $\varphi(x) = \sin x$

$f(\alpha) = \frac{da(\varphi(\alpha))}{d\alpha} = \cos\alpha$ 的极小值点为

$\alpha = 2k\pi + \pi \quad (k \in \mathbb{Z})$

此时 $f(\alpha)$ 取极小值 -1，代入有

$$t_b = -\frac{1}{f(\alpha_0)} = 1.$$

⑥ $a(u) = u^2$. $\varphi(x) = \frac{1}{1+x^2}$

$$a(\varphi(\alpha)) = \frac{1}{(1+\alpha^2)^2} \qquad f(\alpha) = \frac{d}{d\alpha}\left(\frac{1}{(1+\alpha^2)^2}\right)$$

$$= -\frac{4\alpha}{(1+\alpha^2)^3} \qquad \text{则有 } f'(\alpha) = \frac{-4(1+\alpha^2)^3 - 3(1+\alpha^2)^2 \cdot 2\alpha(-4\alpha)}{(1+\alpha^2)^6}$$

$$= \frac{-4(1+\alpha^2) + 24\alpha^2}{(1+\alpha^2)^6} = \frac{20\alpha^2 - 4}{(1+\alpha^2)^6}.$$

从而有 $\alpha_0 = \pm\frac{\sqrt{5}}{5}$. $f(\alpha_0) = \mp\frac{100\sqrt{5}}{216}$.

当 $\alpha_0 = \frac{\sqrt{5}}{5}$ 时 $-\frac{100\sqrt{5}}{216}$ 为其极小值

$$t_b = -\frac{1}{f(\alpha_0)} = \frac{\sqrt{5}}{4}\cdot\left(\frac{6}{5}\right)^3$$

最早爆破地点.

$$x_b = \alpha_0 + a(\varphi(\alpha_0))\, t_b = \frac{\sqrt{5}}{2}.$$

第七次习题课

Goursat 问题

$$\begin{cases} u_{tt}-a^2u_{xx}=0 \quad t>0 \quad -at<x<at & ① \\ u|_{x-at=0}=\varphi(x) & ② \\ u|_{x+at=0}=\psi(x) & ③ \\ \varphi(0)=\psi(0) & ④ \end{cases}$$

解：假设解具有形式

$$u(x,t)=F(x-at)+G(x+at) \quad ⑤$$

由 ② $\Rightarrow$ $\varphi(x)=F(0)+G(2x)$

$\Rightarrow$ $G(x)=\varphi\left(\frac{x}{2}\right)-F(0)$ ⑥

由 ③ $\Rightarrow$ $\psi(x)=F(2x)+G(0)$

$\Rightarrow$ $F(x)=\psi\left(\frac{x}{2}\right)-G(0)$. ⑦

由 ⑤. ⑥ 知 $u(x,t)=F(x-at)+\varphi\left(\frac{x+at}{2}\right)-F(0)$

⑥. ⑦ 知 $\psi(x)=F(2x)+\varphi(0)-F(0)$

从而 $F(x)=\psi\left(\frac{x}{2}\right)-\varphi(0)+F(0)$

$$u(x,t)=\psi\left(\frac{x-at}{2}\right)-\varphi(0)+\varphi\left(\frac{x+at}{2}\right)$$

波动方程的分离变量法

$$\begin{cases} u_{tt}-a^2u_{xx}+2\alpha u_t=0 \\ u|_{t=0}=\varphi(x) \qquad u_t|_{t=0}=\psi(x) \\ u|_{x=0}=0 \quad . \quad u|_{x=l}=0 \end{cases}$$

解：设 $u(x,t)=X(x)T(t)$ 代入

$$XT''-a^2X''T+2\alpha XT'=0$$

有 $$\frac{X''}{X}=\frac{T''+2\alpha T'}{a^2T}=-\lambda$$

$$\Rightarrow\begin{cases}X''+\lambda X=0\\ X(0)=X(l)=0\end{cases}\quad 且\quad T''+2\alpha T'+\lambda a^2T=0$$

对 X 的方程，$\lambda_n=\left(\frac{n\pi}{l}\right)^2\quad X_n=\sin\frac{n\pi}{l}x\quad n=1,2\cdots$

对于 $\lambda=\lambda_n$，解方程 $\mu^2+2\alpha\mu+\lambda_na^2=0$

有根 $\mu=-\alpha\pm\sqrt{\alpha^2-\lambda_na^2}$

于是

$$T_n=\begin{cases}e^{-\alpha t}\left(a_ne^{-\sqrt{\alpha^2-\lambda_na^2}\,t}+b_ne^{\sqrt{\alpha^2-\lambda_na^2}\cdot t}\right) & \left(n<\frac{\alpha l}{a\pi}\right)\\ e^{-\alpha t}(a_n+b_n) & \left(n=\frac{\alpha l}{a\pi}\right)\\ e^{-\alpha t}\left(a_n\cos\left(\sqrt{\lambda_na^2-\alpha^2}\cdot t\right)+b_n\sin\sqrt{\lambda_na^2-\alpha^2}\,t\right) & \left(n>\frac{\alpha l}{a\pi}\right)\end{cases}$$

$$u(x,t)=\sum_{n=1}^{\infty}X_nT_n=\sum_{n=1}^{\infty}T_n\sin\frac{n\pi}{l}x$$

$$u(x,0)=\sum_{n=1}^{\infty}T_n(0)\sin\frac{n\pi}{l}x=\varphi(x)$$

$$u_t(x,0)=\sum_{n=1}^{\infty}T_n'(0)\sin\frac{n\pi}{l}x=\psi(x)$$

定义 $$\tilde{a}_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\,dx$$

$$\tilde{b}_n=\frac{2}{l}\int_0^l\psi(x)\sin\frac{n\pi}{l}x\,dx$$

则有 $$\begin{cases}\varphi(x)=\sum_{n=1}^{\infty}\tilde{a}_n\sin\frac{n\pi}{l}x\\ \psi(x)=\sum_{n=1}^{\infty}\tilde{b}_n\sin\frac{n\pi}{l}x\end{cases}$$

从而
$$\begin{cases} a_n + b_n = \tilde{a}_n \\ -\alpha(a_n+b_n) + \sqrt{\alpha^2-\lambda_n a^2}\,(b_n - a_n) = \tilde{b}_n \end{cases} \quad \left(n < \frac{\alpha l}{a\pi}\right),$$

$$\begin{cases} a_n = \tilde{a}_n \\ -\alpha a_n + b_n = \tilde{b}_n \end{cases} \quad \left(n = \frac{\alpha l}{a\pi}\right)$$

$$\begin{cases} a_n = \tilde{a}_n \\ -\alpha a_n + \sqrt{\lambda_n a^2 - \alpha^2}\, b_n = \tilde{b}_n \end{cases} \quad \left(n > \frac{\alpha l}{a\pi}\right)$$

可解得
$$\begin{cases} a_n = \frac{1}{2}\left(\tilde{a}_n - \frac{1}{\sqrt{\alpha^2-\lambda_n a^2}}(\tilde{a}_n + \tilde{b}_n)\right) \\ b_n = \frac{1}{2}\left(\tilde{a}_n + \frac{1}{\sqrt{\alpha^2-\lambda_n a^2}}(\tilde{a}_n + \tilde{b}_n)\right) \end{cases} \quad n < \frac{\alpha l}{a\pi}$$

$$\begin{cases} a_n = \tilde{a}_n \\ b_n = \alpha \tilde{a}_n + \tilde{b}_n \end{cases} \quad \left(n = \frac{\alpha l}{a\pi}\right)$$

$$\begin{cases} a_n = \tilde{a}_n \\ b_n = \frac{1}{\sqrt{\lambda_n a^2 - \alpha^2}}(\tilde{a}_n + \tilde{b}_n) \end{cases} \quad \left(n > \frac{\alpha l}{a\pi}\right)$$

$$u(x,t) = e^{-\alpha t} \sum_{n<\frac{\alpha l}{a\pi}} \left(a_n e^{-\sqrt{\alpha^2-\lambda_n a^2}\,t} + b_n e^{\sqrt{\alpha^2-\lambda_n a^2}\,t}\right) \sin\frac{n\pi}{l}x$$

$$+ e^{-\alpha t}(a_{n_0} + b_{n_0}) \sin\frac{n_0\pi}{l}x$$

$$+ e^{-\alpha t} \sum_{n>\frac{\alpha l}{a\pi}} \left(a_n \cos\sqrt{\lambda_n a^2-\alpha^2}\,t + b_n \sin\sqrt{\lambda_n a^2-\alpha^2}\,t\right) \sin\frac{n\pi}{l}x$$

第八次习题课

一、Laplace 方程的分离变量法

例 1：
$$\begin{cases} \Delta u = 0 & 1<r<2 \\ u_r|_{r=1} = \sin\theta \\ u_r|_{r=2} = 0 \end{cases}$$

解：设 $u = R(r)\Theta(\theta)$.

则 $\Delta u = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$

$$\Rightarrow \frac{r^2R''+rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

$$\Rightarrow \begin{cases} \Theta'' + \lambda\Theta = 0 \\ r^2R''+rR'-\lambda R = 0 \end{cases}$$

$\Theta_n = a_n\cos n\theta + b_n\sin n\theta$ 易见

$$u = \sum_{n=1}^{\infty}(C_nr^n + D_nr^{-n})(a_n\cos n\theta + b_n\sin n\theta) + a\ln r + b$$

由边界条件知可设 $u = \left(cr + \frac{d}{r}\right)\sin\theta + a\ln r + b$.

$$u_r = \left(c - \frac{d}{r^2}\right)\sin\theta + \frac{a}{r}.$$

$$\begin{cases} u_r|_{r=1} = (c-d)\sin\theta + a = \sin\theta \\ u_r|_{r=2} = \left(c-\frac{d}{4}\right)\sin\theta + \frac{a}{2} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = 0 \\ c = -\frac{1}{3} \\ d = -\frac{4}{3} \end{cases} \Rightarrow u = \left(-\frac{r}{3} - \frac{4}{3}\cdot\frac{1}{r}\right)\sin\theta + b$$
$$= -\frac{x}{3} - \frac{4}{3}\frac{y}{x^2+y^2} + b.$$

例 2：
$$\begin{cases} \Delta u = xy & \text{in } r<a \\ u = 0 & \text{on } r=a \end{cases}.$$

解：令 $v = u - \frac{1}{12}(x^3y + xy^3)$

则 $\Delta v = \Delta u - xy = 0$

$$\begin{cases} \Delta v = 0 & \text{in } r < a \\ v = -\frac{1}{12}(x^3y + xy^3) & \text{on } r = a . \end{cases}$$

即 $v = -\frac{a^4}{12}(\cos^3\theta \sin\theta + \sin^3\theta\cos\theta) = -\frac{a^4}{12}\sin\theta\cos\theta$

从而有 $v = -\frac{a^2}{12}r^2\sin\theta\cos\theta \quad \text{in } r < a$

即 $v = -\frac{a^2}{12}xy . \qquad u = \frac{1}{12}(x^3y + xy^3) - \frac{a^2}{12}xy .$

例3：
$$\begin{cases} \Delta u = A + Bxy & a < r < b \\ u|_{r=a} = u|_{r=b} = 0 \end{cases}$$

解：令 $v = u - \frac{A}{4}(x^2+y^2) - \frac{B}{12}(x^3y + xy^3)$

则
$$\begin{cases} \Delta v = 0 & a < r < b \\ v|_{r=a} = -\frac{A}{4}a^2 - \frac{B}{12}a^2 xy = -\frac{Aa^2}{4} - \frac{Ba^4}{12}\cos\theta\sin\theta \\ v|_{r=b} = -\frac{A}{4}b^2 - \frac{B}{12}b^2 xy = -\frac{Ab^2}{4} - \frac{Bb^4}{12}\cos\theta\sin\theta \end{cases}$$

由边界条件，可设 $v = c_1\ln r + c_2 + (c_3 r^2 + c_4 r^{-2})\sin 2\theta .$

$v|_{r=a} = c_1\ln a + c_2 + 2(c_3 a^2 + \frac{c_4}{a^2})\sin\theta\cos\theta$

$v|_{r=b} = c_1\ln b + c_2 + 2(c_3 b^2 + \frac{c_4}{b^2})\sin\theta\cos\theta$

解得
$$\begin{cases} c_1 = \frac{A}{4}\frac{b^2\ln a - a^2\ln b}{\ln b - \ln a} \\ c_2 = \frac{A}{4}\frac{b^2 - a^2}{\ln a - \ln b} \\ c_3 = \frac{B}{24}\frac{b^6 - a^6}{a^4 - b^4} \\ c_4 = \frac{B}{24}\frac{b^2 - a^2}{a^{-4} - b^{-4}} \end{cases}$$

再代入即有 $u = v + \frac{A}{4}(x^2+y^2) + \frac{B}{12}(x^3y + xy^3)$

$= \frac{A}{4}r^2 + \frac{B}{24}r^4\sin 2\theta + c_1 + c_2\ln r + C_3 r^2\sin 2\theta + C_4 r^{-2}\sin 2\theta$

二. Harnack 不等式

$\Delta u=0$, $u\geq 0$ in $B_R(0)$, $u=\varphi$ on $\partial B_R(0)$.

则有 $\dfrac{R-r}{R+r}u(0)\leq u(x,y)\leq \dfrac{R+r}{R-r}u(0)$ $\quad (r=\sqrt{x^2+y^2})$

证明: 由 Poisson 积分公式有

$$u(r\cos\theta, r\sin\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\eta)+r^2}\varphi(R\cos\eta, R\sin\eta)\,d\eta$$

由 $R^2-2Rr\cos(\theta-\eta)+r^2\geq (R-r)^2$ 知.

$$u(x,y)\leq \frac{R^2-r^2}{(R-r)^2}\frac{1}{2\pi}\int_0^{2\pi}\varphi(R\cos\eta, R\sin\eta)d\eta$$

$$=\frac{(R+r)(R-r)}{(R-r)^2}u(0)=\frac{R+r}{R-r}u(0).$$

同理 由 $R^2-2Rr\cos(\theta-\eta)+r^2\geq (R-r)^2$ 知

$u(x,y)\geq \dfrac{R-r}{R+r}u(0)$ 故得证.

一般的.(n维情形) 我们有

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}}u(0)\leq u(\vec{x})\leq \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}}u(0)$$

三. 热方程的分离变量法

例1: $\begin{cases} u_t=a^2u_{xx}+f(x,t) & (0<x<L\quad t>0)\\ u|_{t=0}=\varphi(x) \\ u_x|_{x=0}=0, \quad u_x|_{x=L}=0\end{cases}$

解: 设 $u=X(x)T(t)$. 对齐次方程

$\begin{cases} u_t=a^2u_{xx}\\ u_x|_{x=0}=0 \quad u_x|_{x=L}=0\end{cases}$ 有 $\begin{cases} XT'=a^2X''T\\ X'(0)=0, \ X'(L)=0\end{cases}$

$$\Rightarrow \frac{X''}{X} = \frac{1}{a^2}\frac{T'}{T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, \ X'(L) = 0 \end{cases}$$ 知 $X_n(x) = A_n \cos\frac{n\pi}{L}x$. $n \in N$

设 $u(x,t) = \sum_{n=0}^{\infty} g_n(t)\cos\frac{n\pi}{L}x$.

$$g_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\frac{n\pi}{L}x\,dx \quad n = 1, 2, 3\cdots$$

$$g_0(t) = \frac{1}{L}\int_0^L u(x,t)\,dx$$

设 $f(x,t) = \sum_{n=0}^{\infty} f_n(t)\cos\frac{n\pi}{L}x$

代入有 $g_n'(t) + \left(\frac{an\pi}{L}\right)^2 g_n(t) = f_n(t)$

又由 $\varphi(x) = \sum_{n=0}^{\infty} g_n(0)\cos\frac{n\pi}{L}x$

从而 $g_0(0) = \frac{1}{L}\int_0^L \varphi(x)\,dx$

$$g_n(0) = \frac{2}{L}\int_0^L \varphi(x)\cos\frac{n\pi x}{L}\,dx \quad n \in N^*$$

利用 ODE 可解出 $g_n(t)$ 的表达式 .

代入可得到 u 的表达式 .

例 2 : $$\begin{cases} u_t = u_{xx} + \sin\frac{x}{2} \\ u|_{t=0} = \sin\frac{x}{2} \\ u|_{x=0} = 0 \quad u_x|_{x=\pi} = 0 \end{cases}$$

解 : $X_n(x) = A_n \sin\frac{2n+1}{2}x$

设 $u(x,t) = \sum_{n=0}^{\infty} g_n(t)\sin\frac{2n+1}{2}x$

则 $u|_{t=0} = \sin\frac{x}{2} = \sum_{n=0}^{\infty} g_n(0)\sin\frac{2n+1}{2}x$

从而有 $g_0(0) = 1$. $g_n(0) = 0 \quad (n \in N^*)$

故有 $\begin{cases} g_0'(t)+\frac{1}{4}g_0(t)=1 \\ g_0(0)=1 \end{cases}$ 且 $g_n(t)\equiv 0 \quad (n\in N^*)$

解得 $g_0(t)=-3e^{-\frac{t}{4}}+4$

$u(x,t)=\left(4-3e^{-\frac{t}{4}}\right)\sin\frac{x}{2}$ 为原方程的解.

总结：对方程 $X''+\lambda X=0$，由边界条件

可以有如下四种特征函数系.

(1) $u|_{x=0}=0$，$u|_{x=L}=0$ $\quad \lambda_n=\left(\frac{n\pi}{L}\right)^2 \quad \left\{\sin\frac{n\pi}{L}x\right\}$.

(2) $u_x|_{x=0}=0$，$u_x|_{x=L}=0$ $\quad \lambda_n=\left(\frac{n\pi}{L}\right)^2 \quad \left\{\cos\frac{n\pi}{L}x\right\}$

(3) $u|_{x=0}=0 \quad u_x|_{x=L}=0 \quad \lambda_n=\left[\frac{(2n+1)\pi}{2L}\right]^2 \quad \left\{\sin\frac{(2n+1)\pi}{2L}x\right\}$

(4) $u_x|_{x=0}=0 \quad u|_{x=L}=0 \quad \lambda_n=\left[\frac{(2n+1)\pi}{2L}\right]^2 \quad \left\{\cos\frac{(2n+1)\pi}{2L}x\right\}$.

第九次习题课

引理：$\int_0^t \sin\omega\tau \sin k(t-\tau)\, d\tau$

$$= \begin{cases} -\dfrac{k}{\omega^2-k^2}\sin\omega t + \dfrac{\omega}{\omega^2-k^2}\sin kt & (\omega\neq k) \\ -\dfrac{t}{2}\cos\omega t + \dfrac{1}{2\omega}\sin\omega t & (\omega = k) \end{cases}$$

证明：$\int_0^t \sin\omega\tau \sin k(t-\tau)\, d\tau$

(当 $\omega\neq k$ 时) $= -\frac{1}{2}\int_0^t \cos\left(\omega\tau + k(t-\tau)\right) d\tau$

$+\frac{1}{2}\int_0^t \cos(\omega\tau - k(t-\tau))\, d\tau$

$= -\frac{1}{2}\frac{1}{\omega-k}\sin(\omega\tau + k(t-\tau))\Big|_0^t$

$\quad +\frac{1}{2}\frac{1}{\omega+k}\sin(\omega\tau - k(t-\tau))\Big|_0^t$

$= \frac{-k}{\omega^2-k^2}\sin\omega t + \frac{\omega}{\omega^2-k^2}\sin kt$

当 $\omega = k$ 时 $\int_0^t \sin\omega\tau \sin(\omega(t-\tau))\, d\tau$

$= -\frac{1}{2}\int_0^t \left(\cos\omega t - \cos(2\omega\tau - \omega t)\right) d\tau$

$= -\frac{t}{2}\cos\omega t + \frac{1}{2}\cdot\frac{1}{2\omega}\sin(2\omega\tau - \omega t)\Big|_0^t$

$= -\frac{t}{2}\cos\omega t + \frac{1}{2\omega}\sin\omega t$

以下记

$$f_{\omega,k}(t) = \begin{cases} -\dfrac{k}{\omega^2-k^2}\sin\omega t + \dfrac{\omega}{\omega^2-k^2}\sin kt \\ -\dfrac{t}{2}\cos\omega t + \dfrac{1}{2\omega}\sin\omega t \end{cases}$$

I. 波动方程共振现象

例1：
$$\begin{cases} u_{tt} = u_{xx} + \sin x\cos 5x\sin\omega t \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0 \\ u|_{x=0} = 0, \quad u|_{x=\pi} = 0 \end{cases}$$

求出所有 ω，使得 $\sup\limits_{\Omega} |u| < +\infty$.

解：
$$\begin{cases} v_{tt} = v_{xx} \\ v|_{t=\tau} = 0, \quad v_t|_{t=\tau} = \sin x\cos 5x\sin\omega\tau \\ v|_{x=0} = 0, \quad v|_{x=\pi} = 0 \end{cases}$$

$$v = \sum_{n=1}^{\infty} A_n \sin n(t-\tau)\sin nx$$

$$v_t|_{t=\tau} = \sum_{n=1}^{\infty} nA_n \sin nx = \sin x\cos 5x\sin\omega\tau$$

$$= \frac{1}{2}\sin\omega\tau(\sin 6x - \sin 4x)$$

$$\Rightarrow A_4 = -\frac{1}{8}\sin\omega\tau, \quad A_6 = \frac{1}{12}\sin\omega\tau$$

$$v = -\frac{1}{8}\sin\omega\tau\sin 4(t-\tau)\sin 4x + \frac{1}{12}\sin\omega\tau\sin 6(t-\tau)\sin 6x$$

$$u(x,t) = \int_0^t v(x,t;\tau)\,d\tau \qquad (\text{当 } \omega \neq \pm 4, \pm 6 \text{ 时})$$

$$= -\frac{1}{8}\int_0^t \sin\omega\tau\sin 4(t-\tau)\,d\tau\,\sin 4x + \frac{1}{12}\int_0^t \sin\omega\tau\sin 6(t-\tau)\,d\tau\,\sin 6x$$

$$= -\frac{1}{8}\left[\frac{-4}{\omega^2-16}\sin\omega t + \frac{\omega}{\omega^2-16}\sin 4t\right]\sin 4x$$

$$+ \frac{1}{12}\left[\frac{-6}{\omega^2-36}\sin\omega t + \frac{\omega}{\omega^2-36}\sin 6t\right]\sin 6x$$

显然有 $\sup\limits_{\Omega} |u| < +\infty$ （$\because$ 三角函数有界）

而 $\omega = 4$ 时

$$u(x,t) = -\frac{1}{8}\int_0^t \sin 4\tau \sin 4(t-\tau)\,d\tau \sin 4x$$

$$+\frac{1}{12}\left[\frac{-6}{-20}\sin 4t + \frac{4}{-20}\sin 6t\right]\sin 6x$$

$$= -\frac{1}{8}\left[\frac{t}{2}\cos 4t + \frac{1}{8}\sin 4t\right] + O(1)$$

$$= -\frac{t}{16}\cos 4t + O(1) \quad 无界$$

故此时 $\sup\limits_{Q}|u| = +\infty$. 同理知 $\omega = -4, 6, -6$ 时

有 $\sup\limits_{Q}|u| = +\infty$. 故满足题意的 $\omega \in \mathbb{R} \setminus \{\pm 4, \pm 6\}$.

例 2: 求解
$$\begin{cases} u_{tt} = u_{xx} \\ u|_{t=0} = 0 \quad u_t|_{t=0} = 0 \\ u|_{x=0} = 0 \quad u_x|_{x=l} = A\sin\omega t \end{cases}$$

解: 令 $U = Ax\sin\omega t$ $\quad v \triangleq u - U$.

则
$$\begin{cases} v_{tt} - v_{xx} = -A\omega^2 x\sin\omega t \\ v|_{t=0} = 0, \quad v_t|_{t=0} = -A\omega x \\ v|_{x=0} = 0 \quad v_x|_{x=l} = 0. \end{cases}$$

设 $v = \tilde{v} + \bar{v}$. 其中

$$\begin{cases} \tilde{v}_{tt} = \tilde{v}_{xx} - A\omega^2 x\sin\omega t \\ \tilde{v}|_{t=0} = 0 \quad \tilde{v}_t|_{t=0} = 0 \\ \tilde{v}|_{x=0} = 0 \quad \tilde{v}_x|_{x=l} = 0 \end{cases} \qquad \begin{cases} \bar{v}_{tt} = \bar{v}_{xx} \\ \bar{v}|_{t=0} = 0 \quad \bar{v}_t|_{t=0} = -A\omega x \\ \bar{v}|_{x=0} = 0 \quad \bar{v}_x|_{x=l} = 0 \end{cases}$$

先解 $\bar{v}$. 设 $\bar{v} = \sum\limits_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}\,t \sin\sqrt{\lambda_n}\,x$

其中 $\sqrt{\lambda_n} = \left(\frac{2n-1}{2l}\right)\pi$

$$\bar{v}_t\Big|_{t=0} = \sum_{n=1}^{\infty} A_n\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\,x = -A\omega x$$

$$\Rightarrow \sqrt{\lambda_n}\cdot A_n = \frac{2}{l}\int_0^l (-A\omega x)\sin\sqrt{\lambda_n}\,x\,dx$$

$$\Rightarrow A_n = \frac{2A\omega}{\lambda_n^{\frac{3}{2}} l}(-1)^n$$

$$\bar{v} = \sum_{n=1}^{\infty}\frac{2A\omega}{\lambda_n^{\frac{3}{2}} l}(-1)^n \sin\left(\frac{2n-1}{2l}\pi t\right)\sin\left(\frac{2n-1}{2l}\pi x\right)$$

再解 $\tilde{v}$. 考虑
$$\begin{cases} g_{tt} - g_{xx} = 0 \\ g\big|_{t=\tau} = 0, \quad g_t\big|_{t=\tau} = -A\omega^2 x\sin\omega\tau \\ g\big|_{x=0} = 0 \quad g_x\big|_{x=l} = 0 \end{cases}$$

设 $g = \sum_{n=1}^{\infty} B_n \sin\sqrt{\lambda_n}(t-\tau)\sin\sqrt{\lambda_n}\,x$

$$g_t\Big|_{t=0} = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\cdot B_n\sin\sqrt{\lambda_n}\,x = -A\omega^2 x\sin\omega\tau$$

$$\Rightarrow B_n = \frac{2A\omega^2}{L\lambda_n^{\frac{3}{2}}}(-1)^n\sin\omega\tau$$

故 $g = \sum_{n=1}^{\infty}\frac{2A\omega^2}{L\lambda_n^{\frac{3}{2}}}(-1)^n\sin\omega\tau\,\sin\sqrt{\lambda_n}(t-\tau)\sin\sqrt{\lambda_n}\,x$

$$\tilde{v} = \sum_{n=1}^{\infty}\frac{2A\omega^2}{L\lambda_n^{\frac{3}{2}}}(-1)^n f_{\omega,\sqrt{\lambda_n}}(t)\sin\sqrt{\lambda_n}\,x.$$

从而 $v = \tilde{v} + \bar{v}$. $u = \tilde{v} + \bar{v} + U$ 为解的表达式.

即 $u = \sum_{n=1}^{\infty}\frac{2A\omega^2}{L\lambda_n^{\frac{3}{2}}}(-1)^n f_{\omega,\sqrt{\lambda_n}}(t)\sin\sqrt{\lambda_n}\,x$

$$+\sum_{n=1}^{\infty}\frac{2A\omega}{\lambda_n^{\frac{3}{2}} l}(-1)^n\sin\sqrt{\lambda_n}\,t\,\sin\sqrt{\lambda_n}\,x + Ax\sin\omega t$$

Ⅱ. 变分法（泛函极小问题）

例 1：$\Omega \triangleq \{x\in\mathbb{R}^3 \mid 1<|x|<2\}$.

$$I[w] \triangleq \int_\Omega \left(|\nabla w|^2+2w\right)dx+\int_{|x|=1} w^2\,ds$$

求 $\inf\limits_{w\in M} I[w]$. 其中 $M=\{w \mid w=0 \text{ on } |x|=2\}$.

解：(1) $I[u+tv]=\int_\Omega |\nabla u+t\nabla v|^2dx$

$+\int_\Omega 2(u+tv)\,dx+\int_{|x|=1}(u+tv)^2\,ds$

$$0=\left.\frac{dI}{dt}\right|_{t=0}=2\int_\Omega \nabla u\cdot\nabla v+2\int_\Omega v\,dx+2\int_{|x|=1}uv\,ds$$

$$=2\int_{\partial\Omega}\frac{\partial u}{\partial n}v\,ds-2\int_\Omega(\Delta u\cdot v-v)\,dx+2\int_{|x|=1}uv\,ds$$

$$=2\int_{|x|=2}\frac{\partial u}{\partial n}v\,ds-2\int_{|x|=1}\frac{\partial u}{\partial n}v\,ds$$

$$+2\int_\Omega v(1-\Delta u)\,dx+2\int_{|x|=1}uv\,ds$$

$$=2\int_\Omega v(1-\Delta u)\,dx+2\int_{|x|=1}\left(u-\frac{\partial u}{\partial n}\right)v\,ds$$

(2)
$$\Rightarrow\begin{cases}\Delta u=1 & \text{in } \Omega\\ u=\dfrac{\partial u}{\partial n} & \text{on } |x|=1\\ u=0 & \text{on } |x|=2\end{cases}\qquad 由\ \left.\frac{\partial u}{\partial n}\right|_{|x|=1}=-\left.\frac{\partial u}{\partial r}\right|_{r=1}$$

$$\Rightarrow\begin{cases}\Delta u=1 & \text{in } \Omega\\ u+\dfrac{\partial u}{\partial r}=0 & \text{on } r=1\\ u=0 & \text{on } r=2\end{cases}$$

令 $u = \frac{A}{r} + B + \frac{r^2}{6}$. 代入有

$$\begin{cases} -A + \frac{1}{3} + A + B + \frac{1}{6} = 0 \\ \frac{A}{2} + B + \frac{2}{3} = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{3} \\ B = -\frac{1}{2} \end{cases}.$$

$$u = -\frac{1}{3r} - \frac{1}{2} + \frac{r^2}{6} \qquad u_r = \frac{1}{3r^2} + \frac{r}{3}.$$

(3) $$I[u] = \int_\Omega \left(|\nabla u|^2 + 2u \right) dx + \int_{|x|=1} u^2 dx$$

$$\int_\Omega \left(|\nabla u|^2 + 2u \right) dx = \int_\Omega \left(u_r^2 + 2u \right) dx$$

$$= \int_1^2 \left[\left(\frac{1}{3r^2} + \frac{r}{3} \right)^2 + 2\left(-\frac{1}{3r} - \frac{1}{2} + \frac{r^2}{6} \right) \right] 4\pi r^2 dr$$

$$= 4\pi \int_1^2 \left(\frac{1}{9r^2} - \frac{4}{9} r + \frac{4}{9} r^4 - r^2 \right) dr$$

$$= 4\pi \left(\frac{1}{18} - \frac{2}{3} + \frac{124}{45} - \frac{7}{3} \right) = -\frac{34}{45}\pi$$

而 $$\int_{|x|=1} u^2 dx = u\big|_{r=1}^2 \cdot 4\pi = \frac{16}{9}\pi.$$

故 $$I[u] = \frac{16}{9}\pi - \frac{34}{45}\pi = \frac{46}{45}\pi.$$

例2: $\Omega = B_1^2(0)$, $I[w] \triangleq \int_\Omega |\nabla w|^2 dx$

求 $\inf\limits_{w \in M} I[w]$ 其中 $M \triangleq \{ w \mid w = x_2^2 \text{ on } \partial B_1^2(0) \}$

解: $$I[u+tv] = \int_\Omega |\nabla u|^2 + 2t \nabla u \cdot \nabla v + |\nabla v|^2 dx$$

$$0 = \frac{dI}{dt}\bigg|_{t=0} = 2\int_\Omega \nabla u \cdot \nabla v \, dx = -2\int_\Omega \Delta u \cdot v \, dx$$

$\Rightarrow \Delta u = 0$ in Ω.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \sin^2\theta & \text{on } \partial\Omega \end{cases}$$ 猜得 $u = \dfrac{1 - r^2\cos 2\theta}{2}$

即 $u = \dfrac{1}{2} - \dfrac{x^2}{2} + \dfrac{y^2}{2}$ $\quad \nabla u = (-x, y)$.

$|\nabla u|^2 = x^2 + y^2 = r^2$.

$I[u] = \int_0^1 \int_0^{2\pi} r^2 \cdot r \, d\theta \, dr = \dfrac{\pi}{2}$.

Ⅲ 泛函极小化序列

设 $\Omega \subset \mathbb{R}^n$ 为有界区域，且 $\Omega \subset I[-R, R]$.

其中 $I[-R, R] = \{(x_1, \cdots x_n) \mid |x_i| \leq R, \ i = 1, 2 \cdots n\}$.

$$I[w] \triangleq \int_\Omega |\nabla w|^2 + 2wf \, dx$$

$\mu \triangleq \inf\limits_{w \in M} I[w] \qquad M = \{w \mid w = \varphi \text{ on } \partial\Omega\}$.

则存在 $u \in M$, s.t. $I[u] = \mu$.

证明：$I\left[\dfrac{u_m + u_n}{2}\right] = \int_\Omega \dfrac{1}{4}|\nabla u_m + \nabla u_n|^2 + (u_m + u_n) f \, dx$

故 $\dfrac{1}{4}\int_\Omega |\nabla u_m - \nabla u_n|^2 dx + I\left[\dfrac{u_m + u_n}{2}\right]$

$= \int_\Omega \left(\dfrac{1}{2}|\nabla u_m|^2 + \dfrac{1}{2}|\nabla u_n|^2 + u_m f + u_n f\right) dx$

$= \dfrac{1}{2} I[u_m] + \dfrac{1}{2} I[u_n] < \mu + \dfrac{1}{2}\left(\dfrac{1}{m} + \dfrac{1}{n}\right)$

而 $I\left[\dfrac{u_m + u_n}{2}\right] \geq \mu$. 故得到

$$\|\nabla u_m - \nabla u_n\|_{L^2}^2 < 2\left(\frac{1}{m} + \frac{1}{n}\right)$$

从而有 $\lim\limits_{m, n \to \infty} \|\nabla u_m - \nabla u_n\|_{L^2} = 0 \quad \cdots (*)$

方法一：由上式知 $\|u_m-u_n\|_{H_0^1(\Omega)}\to 0$.

从而 $\{u_n\}$ 为 $H_0^1(\Omega)$ 中柯西列，故 $u_n\to u$ in $H_0^1(\Omega)$

方法二（本质上与方法一相同）

引入 Poincare 不等式：

设 $u=0$ on $\partial\Omega$ （Ω 为有界区域）

则 $\exists\ C$ 只依赖于 Ω. 有下式成立.

$$\int_\Omega |u(x)|^2\,dx \le C\int_\Omega |\nabla u(x)|^2\,dx$$

证明：将 u 扩充为定义在 $I[-R,R]$ 中函数

即将 $I\setminus\Omega$ 中点上的值取为 0.

则 $$u(x)=\int_{-\infty}^{x_1}\frac{\partial u}{\partial x_1}\,dx_1=\int_{-R}^{x_1}\frac{\partial u}{\partial x_1}\,dx_1$$

$$|u(x)|^2=\left(\int_{-R}^{x_1}\frac{\partial u}{\partial x_1}(x_1\cdots x_n)\,dx_1\right)^2$$

$$\le\int_{-R}^{x_1}\left|\frac{\partial u}{\partial x_1}\right|^2dx_1\int_{-R}^{x_1}1\,dx_1$$

$$\le\int_{-R}^{R}\left|\frac{\partial u}{\partial x_1}\right|^2dx_1\cdot 2R$$

从而有 $$\int_\Omega|u(x)|^2dx=\int_{-R}^{R}\cdots\int_{-R}^{R}|u(x)|^2dx_1\cdots dx_n$$

$$\le\int_{-R}^{R}\cdots\int_{-R}^{R}\left(2R\int_{-R}^{R}\left|\frac{\partial u}{\partial x_1}(x_1\cdots x_n)\right|^2dx_1\right)dx_1\cdots dx_n$$

$$=\int_{-R}^{R}\cdots\int_{-R}^{R}\left(2R\int_{-R}^{R}|u_1(y.x_2\cdots x_n)|^2dy\right)dx_1\cdots dx_n$$

$$=\underbrace{\int_{-R}^{R}\cdots\int_{-R}^{R}}_{n-1\text{个}}\left(4R^2\int_{-R}^{R}|u_1(y.x_2\cdots x_n)|^2dy\right)dx_2\cdots dx_n$$

$$= 4R^2 \int_{-R}^{R} \cdots \int_{-R}^{R} |u_1(x_1 \cdots x_n)|^2 \, dx_1 \cdots dx_n$$

$$\leq 4R^2 \int_{-R}^{R} \cdots \int_{-R}^{R} |\nabla u|^2 \, dx_1 \cdots dx_n$$

$$= 4R^2 \int_{\Omega} |\nabla u|^2 \, dx$$

即定理中的 C 可取为 $4R^2$，只依赖于 Ω 的大小.

由 Poincare 不等式并结合 $(u_m - u_n)\big|_{\partial\Omega} = 0$

知 $\|u_m - u_n\|_{L^2} \leq \tilde{C} \|\nabla u_m - \nabla u_n\|_{L^2} \to 0$.

从而结合 L^2 完备性知 $u_n \to u$.

故 $\exists u$ s.t. $\lim\limits_{n\to\infty} u_n = u$.

从而 $\lim\limits_{n\to\infty} I[u_n] = \mu \Rightarrow I[u] = \mu$.

即极小值点是存在的（即可以取到极小）

注：称求 $\inf\limits_{w\in M} I[w]$ 为微分方程问题

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$ 的变分提法.

上面详细地讨论了变分提法解的存在性问题

第十次习题课

波动方程能量法

初值问题 $\begin{cases} u_{tt} - a^2 \Delta u = 0 & \text{in } \Omega \times [0, +\infty) \\ \alpha \dfrac{\partial u}{\partial n} + \sigma u = 0 & \text{on } \partial\Omega \times [0, +\infty) \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x) \end{cases}$

同乘 u_t 两边在 Ω 上积分有

$$\int_\Omega (u_{tt} - a^2 \Delta u) u_t \, dx = 0$$

又 $\displaystyle\int_\Omega u_t u_{tt} - a^2 \Delta u \cdot u_t \, dx = \frac{d}{dt}\left(\frac{1}{2}\int_\Omega u_t^2 dx\right)$

$$- a^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial n} ds + a^2 \int_\Omega \nabla u \cdot \nabla u_t \, dx$$

$$= \frac{d}{dt}\left(\frac{1}{2}\int_\Omega u_t^2 + a^2 |\nabla u|^2 dx\right) - a^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial n} ds \quad \cdots ①$$

当 $\alpha = 0$, $\sigma \neq 0$ 时 $u|_{\partial\Omega \times [0,+\infty)} = 0 \Rightarrow u_t|_{\partial\Omega} = 0$

① 式可化为 $\displaystyle\frac{d}{dt}\left(\frac{1}{2}\int_\Omega (u_t^2 + a^2 |\nabla u|^2) dx\right) = 0$

定义 $E(t) \triangleq \dfrac{1}{2}\displaystyle\int_\Omega u_t^2 + a^2 |\nabla u|^2 dx$ 则 $\dfrac{dE(t)}{dt} = 0$. $E(t) \equiv C$

当 $\alpha \neq 0$ 时 $\left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = -\dfrac{\sigma}{\alpha} u\Big|_{\partial\Omega \times [0,+\infty)}$

① 式可化为 $\displaystyle\frac{d}{dt}\left(\frac{1}{2}\int_\Omega (u_t^2 + a^2 |\nabla u|^2) dx\right)$

$$+ \frac{a^2 \sigma}{\alpha} \int_{\partial\Omega} u_t \, ds = 0.$$

即 $\displaystyle\frac{d}{dt}\left(\frac{1}{2}\int_\Omega (u_t^2 + a^2 |\nabla u|^2) dx + \frac{a^2\sigma}{2\alpha}\int_{\partial\Omega} u^2 ds\right) = 0$

定义 $E(t) \triangleq \dfrac{1}{2}\displaystyle\int_\Omega (u_t^2 + a^2 |\nabla u|^2) dx + \frac{a^2 \sigma}{2\alpha}\int_{\partial\Omega} u^2 ds$

则 $E(t) \equiv C$ 　　即能量是守恒的

例1：设 u 为如下初值问题的解

$$\begin{cases} u_{tt} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(p_i(x) u_{x_i}\right) + c^2 u = 0 & \text{in } \Omega \times (0,+\infty) \\ u\big|_{\partial\Omega \times [0,+\infty)} = 0 \\ u\big|_{t=0} = \varphi(x), \quad u_t\big|_{t=0} = \psi(x) \end{cases}$$

其中 $\Omega \subset \mathbb{R}^3$ 为具有光滑区域 $p_1, p_2, p_3 \geq a^2 > 0$ 为适当光滑函数，$a, c \neq 0$，求证 $\exists M > 0$，s.t.

$$\iiint_\Omega \left(u^2 + |\nabla u|^2 + u_t^2\right) dx_1 dx_2 dx_3$$

$$\leq M \iiint_\Omega \left(\varphi^2 + |\nabla\varphi|^2 + \psi^2\right) dx_1 dx_2 dx_3 \qquad \forall t > 0.$$

证明：由 $\int_\Omega u_t u_{tt} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(p_i(x) u_i\right) u_t + c^2 u u_t \, dx = 0$

$$\Rightarrow \frac{d}{dt} \iiint_\Omega \frac{1}{2}\left(u_t^2 + p_1 u_1^2 + p_2 u_2^2 + p_3 u_3^2 + c^2 u^2\right) dx\,dy\,dz \leq 0.$$

从而 $\frac{1}{2}\int_\Omega \left(u_t^2 + p_1 u_1^2 + p_2 u_2^2 + p_3 u_3^2 + c^2 u^2\right) dx\,dy\,dz$

$$\leq \frac{1}{2}\int_\Omega \left(\psi^2 + p_1 \varphi_1^2 + p_2 \varphi_2^2 + p_3 \varphi_3^2 + c^2 \varphi^2\right) dx\,dy\,dz$$

$$\leq M \int_\Omega \left(\psi^2 + |\nabla\varphi|^2 + \varphi^2\right) dx\,dy\,dz$$

例2：$\Omega \subset \mathbb{R}^3$ 为有界区域，边界由 T_0，T_1 两部分组成

u 为如下初值问题的解

$$\begin{cases} u_{tt} - a^2 \Delta u = 0 & \text{in } \Omega \times (0,+\infty) \\ u\big|_{T_0} = 0 \qquad \left.\frac{\partial u}{\partial n} + \sigma \frac{\partial u}{\partial t}\right|_{T_1} = 0 & (\sigma > 0) \\ u\big|_{t=0} = \varphi(x,y,z) \qquad u_t\big|_{t=0} = \psi(x,y,z) \end{cases}$$

定义 $E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + a^2 |\nabla u|^2\right) dx\,dy\,dz$　求证 $\frac{dE}{dt} \leq 0$.

并由此说明上述问题解的唯一性

证明： $\frac{dE}{dt} = \int_\Omega (u_t u_{tt} + a^2 \nabla u \nabla u_t) dxdydz$

$= \int_\Omega u_t u_{tt} dxdydz + a^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial n} ds - a^2 \int_\Omega u_t \Delta u \, dxdydz$

$= \int_\Omega u_t (u_{tt} - a^2 \Delta u) dxdydz + a^2 \int_{T_1} u_t (-\sigma \frac{\partial u}{\partial t}) ds$

$+ a^2 \int_{T_0} u_t \frac{\partial u}{\partial n} ds = -a^2 \sigma \int_{T_1} u_t^2 ds \le 0$

故有 $\frac{dE(t)}{dt} \le 0$，即 $E(t)$ 单调减少.

解的唯一性等价于 $\begin{cases} u_{tt} - a^2 \Delta u = 0 \\ u|_{T_0} = 0 \quad \frac{\partial u}{\partial n} + \sigma \frac{\partial u}{\partial t} \Big|_{T_1} = 0 \\ u|_{t=0} = 0 \quad u_t|_{t=0} = 0 \end{cases}$ 只有零解.

由 $E(t) \downarrow$，$E(0) = 0$，知 $E(t) \equiv 0$，从而

$u \equiv const$．结合 $u|_{t=0} = 0$ 知 $u \equiv 0$.

热方程的初边值问题

例1： $\begin{cases} u_t - a^2 \Delta u = 0 \\ \alpha \frac{\partial u}{\partial n} + \sigma u \Big|_\Sigma = 0 \\ u|_{t=0} = \varphi(x,y,z) \end{cases}$

$E_0(t) \triangleq \frac{1}{2} \int_\Omega u^2 \, dxdydz$ 则 $\frac{dE_0}{dt} \le 0$.

即 $E_0(t) \downarrow$．这样有 $\|u(\cdot,t)\|_{L^2(\Omega)} \le \|\varphi\|_{L^2(\Omega)}$

证明： $\int_\Omega u u_t \, dxdydz - a^2 \int_\Omega u \Delta u \, dxdydz = 0$

而 $\int_\Omega u u_t \, dxdydz = \frac{1}{2} \frac{d}{dt} \int_\Omega u^2 \, dxdydz = \frac{dE_0(t)}{dt}$

$\int_\Omega u \Delta u \, dxdydz = \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds - \int_\Omega |\nabla u|^2 dxdydz$

故有 $\frac{dE_0}{dt} - a^2\int_{\partial\Omega} u\frac{\partial u}{\partial n}ds + a^2\int_{\Omega}|\nabla u|^2 dxdydz = 0$

$\alpha=0$ 时 $u|_{\Sigma}=0$ $\frac{dE_0}{dt} = -a^2\int_{\Omega}|\nabla u|^2 dxdydz \le 0$

$\alpha\ne 0$ 时 $\frac{\partial u}{\partial n}\Big|_{\Sigma} = -\frac{\sigma}{\alpha}u\Big|_{\Sigma}$

$$\int_{\partial\Omega} u\cdot\frac{\partial u}{\partial n}ds = -\frac{\sigma}{\alpha}\int_{\partial\Omega}u^2 ds \le 0$$

$$\frac{dE_0}{dt} = -\frac{\sigma}{\alpha}a^2\int_{\partial\Omega}u^2 ds - a^2\int_{\Omega}|\nabla u|^2 dxdydz \le 0$$

由 $\|u(\cdot,t)\|_{L^2(\Omega)} \le \|\varphi\|_{L^2(\Omega)}$ $\forall t>0$.

即知原方程解的唯一性

例2：$\begin{cases} u_{tt} - a^2u_{xx} = 0 \\ u_x - \sigma u|_{x=0} = h(t) \quad u_x+\sigma u|_{x=l} = g(t) \\ u|_{t=0} = \varphi(x) \end{cases}$

其中 $\sigma>0$ 为常数，试证明解 $u(x,t)$ 在下述意义下连续依赖于初边值：$\forall\varepsilon>0$，$\exists\delta>0$ s.t. 当

$\int_0^l \varphi^2(x)dx + \int_0^T[h^2(t)+g^2(t)]dt<\delta$ 时

有 $\int_0^l u^2(x,t)dx<\varepsilon$ $(\forall t\in[0,T])$

证明：$\int_0^l u_t u\,dx = a^2\int_0^l u_{xx}u\,dx$

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2dx\right) - a^2u_xu\Big|_0^l + a^2\int_0^l u_x^2dx = 0$$

注意到 $u_x-\sigma u\,\Big|_{x=0} = h(t)$ $u_x+\sigma u\,\Big|_{x=l} = g(t)$

由上式得 $\frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2dx\right) - a^2g(t)u(l,t) + a^2\sigma u^2(l,t)$

$+a^2h(t)u(0,t) + a^2\sigma u^2(0,t) + a^2\int_0^l u_x^2dx = 0$

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2 dx\right) \le a^2 g(t) u(l,t) - a^2\sigma u^2(l,t)$$

$$- a^2 h(t) u(0,t) - a^2 \sigma u^2(0,t)$$

由 $|g(t) u(l,t)| \le \sigma u^2(l,t) + \frac{1}{4\sigma} g^2(t)$

$|h(t) u(0,t)| \le \sigma u^2(0,t) + \frac{1}{4\sigma} h^2(t)$

所以 $\frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2 dx\right) \le \frac{a^2}{4\sigma}\left[h^2(t) + g^2(t)\right]$

$$\Rightarrow \frac{1}{2}\int_0^l u^2 dx \le \frac{1}{2}\int_0^l \varphi^2(x) dx + \frac{a^2}{4\sigma}\int_0^t \left[h^2(\tau) + g^2(\tau)\right] d\tau$$

$$\int_0^l u^2 dx \le \int_0^l \varphi^2(x) dx + \frac{a^2}{2\sigma}\int_0^t \left[h^2(\tau) + g^2(\tau)\right] d\tau$$

$$\le \max\left\{1, \frac{a^2}{2\sigma}\right\}\left(\int_0^l \varphi^2(x) dx + \int_0^T \left[h^2(\tau) + g^2(\tau)\right] d\tau\right)$$

例3：设 u 为下述问题之解

$$\begin{cases} u_t - u_{xx} + cu = 0 & 0 < x < l,\ t > 0 \\ u|_{x=0} = 0, & u_x + \sigma u|_{x=l} = 0 \\ u|_{t=0} = \varphi(x) \end{cases}$$

其中 $\varphi \in C^1$，$\varphi(0) = 0$，求证 $\lim\limits_{t\to\infty}\int_0^l u^2(x,t) dx = 0$

证明：由 $\int_0^l (u_t - u_{xx} + cu) u \, dx = 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2 dx\right) - u u_x\Big|_0^l + \int_0^l u_x^2 dx + c\int_0^l u^2 dx = 0$$

即 $\frac{d}{dt}\left(\int_0^l u^2 dx\right) + 2\sigma u^2(l,t) + 2\int_0^l (u_x^2 + cu^2) dx = 0$

$$\therefore \frac{d}{dt}\left(\int_0^l u^2 dx\right) + 2c\int_0^l u^2 dx \le 0$$

$$\Rightarrow \int_0^l u^2 dx \le e^{-2ct}\int_0^l \varphi^2(x) dx \quad \forall t > 0$$

令 $t \to +\infty$ 得 $\lim_{t\to\infty} \int_0^l u^2 dx = 0$.

例4: 设 u 为 $\begin{cases} u_t - u_{xx} = 0 & 0<x<l, \ t>0 \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = \varphi(x) \end{cases}$.

其中 $\varphi \in C^1$, $\varphi(0) = \varphi(L) = 0$. 求证

$$\int_0^L u^2(x,t)\, dx \le e^{-ct} \int_0^l \varphi^2(x)\, dx \qquad (c>0).$$

证明: 由 $\int_0^l (u_t - u_{xx})\, u\, dx = 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2 dx\right) - uu_x \Big|_0^L + \int_0^l u_x^2 dx = 0.$$

即 $\frac{d}{dt}\int_0^l u^2 dx = -2\int_0^l u_x^2 dx$.

由 Poincare 不等式得 $c\int_0^l u^2 dx \le 2\int_\Omega u_x^2 dx$

$\therefore \frac{d}{dt}\int_0^l u^2 dx \le -c\int_0^l u^2 dx$ 解之可得

$$\int_0^l u^2(x,t)\, dx \le e^{-ct}\int_0^l \varphi^2(x)\, dx$$

本题也可以利用分离变量法求得解为

$$u(x,t) = \sum_{k=1}^{+\infty} A_k \sin\frac{k\pi x}{L} e^{-\frac{k^2\pi^2}{l^2}t}$$

其中 $A_k = \frac{2}{L}\int_0^l \varphi(x)\sin\frac{k\pi x}{L}\, dx$. 从而由 Parseval 等式

$$\int_0^l u^2(x,t)\, dx = \frac{L}{2}\sum_{k=1}^{+\infty} A_k^2 e^{-\frac{2k^2\pi^2}{l^2}t}$$

$\le e^{-\frac{2\pi^2}{l^2}t}\left(\frac{L}{2}\sum_{k=1}^{+\infty} A_k^2\right)$. 又 $\int_0^l \varphi^2(x)\, dx = \frac{L}{2}\sum_{k=1}^{+\infty} A_k^2$

从而 $\int_0^l u^2 dx \le e^{-\frac{2\pi^2}{L^2}t}\int_0^l \varphi^2(x)\, dx$

第十一次习题课

I. 热方程分离变量法

$$\begin{cases} u_t - a^2 u_{xx} = 0, & (t>0,\ 0<x<L) \\ u|_{t=0} = \varphi(x) & u(0,t) = 0 \\ u_x(L,t) + h u(L,t) = 0, & h>0 \text{ 为常数} \end{cases}$$

解: $u(x,t) = X(x)T(t)$，代入得

$$\begin{cases} T' + \lambda a^2 T = 0 \\ X'' + \lambda X = 0 \end{cases} \Longrightarrow \begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X'(L) + hX(L) = 0 \end{cases}$$

易见 $\lambda > 0$　$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

代入边界条件可解得　$\lambda_k = \left(\frac{\gamma_k}{L}\right)^2$.

其中 γ_k 为 $\tan x = -\frac{x}{hL}$ 的所有正根

从而有　$X_k(x) = B_k \sin\frac{\gamma_k}{L}x$　代入可解出

$T_k(t) = C_k e^{-a^2\lambda_k t}$

从而　$u_k(x,t) = A_k \sin\sqrt{\lambda_k}x\, e^{-a^2\lambda_k t}$

由叠加原理即可构造级数解　$u = \sum_{k=1}^{\infty} u_k(x,t)$

由 $u|_{t=0} = \sum_{k=1}^{\infty} A_k \sin\sqrt{\lambda_k}x = \varphi(x)$

$$\Longrightarrow A_k = \frac{2}{L}\int_0^L \varphi(x)\sin\sqrt{\lambda_k}x\,dx$$

II. 热方程 Cauchy 问题的求解

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) \\ u|_{t=0} = \varphi(x) \end{cases}$$

解: 先考虑齐次方程 $\begin{cases} u_t - a^2 u_{xx} = 0 \\ u|_{t=0} = \varphi(x) \end{cases}$

作 Fourier 变换 $\begin{cases} \hat{u}_t - a^2\xi^2\hat{u} = 0 \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi) \end{cases}$.

$$\Rightarrow \hat{u}(\xi,t) = \hat{\varphi}(\xi)\, \widehat{e^{-a^2\xi^2 t}}$$

这样 $u(x,t) = \varphi(x) * \left(\widecheck{e^{-a^2\xi^2 t}}\right)$

易见 $\widecheck{e^{-a^2\xi^2 t}} = \dfrac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$

从而 $u(x,t) = \dfrac{1}{2a\sqrt{\pi t}} \displaystyle\int_{-\infty}^{+\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi$

再来看 $\begin{cases} u_t - a^2 u_{xx} = f(x,t) \\ u|_{t=0} = 0 \end{cases}$.

由齐次化原理 $u(x,t) = \displaystyle\int_0^t w(x,t;\tau)\, d\tau$.

其中 $w(x,t;\tau) = \dfrac{1}{2a\sqrt{\pi(t-\tau)}} \displaystyle\int_{-\infty}^{+\infty} f(\xi,\tau) e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi$

从而原方程的解为

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi$$
$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \int_{-\infty}^{+\infty} \frac{f(\xi,\tau)}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi\, d\tau$$

Ⅲ. 波动方程解的衰减.

由 Kirchhoff 公式得

$$u(x,t) = \frac{1}{4\pi a^2 t} \int_{S_{at}} \psi(y)\, dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t} \int_{S_{at}} \varphi(y)\, dS_y\right)$$

$$= \frac{1}{4\pi a^2 t} \int_{S_{at}} \psi(y)\, dS_y + \frac{\partial}{\partial t}\left(\frac{t}{4\pi} \int_{\partial B_1} \varphi(x+at\xi)\, dS_\xi\right)$$

$$= \frac{1}{4\pi a^2 t} \int_{S_{at}} \psi\, dS + \frac{1}{4\pi} \int_{\partial B_1(x)} \left(\varphi + \sum_i ta\xi_i \varphi_i(x+at\xi)\right) dS_\xi$$

$$= \frac{1}{4\pi a^2 t^2} \int_{S_{at}} (t\psi + \varphi + at\nabla\varphi)\, dS$$

由 $\varphi, \psi \in C_c^\infty$ 知 $\exists B_R$ s.t. $\psi, \varphi, \nabla\varphi$ 支在 B_R 上

且 $\exists M$ s.t. $|\psi| < M$, $|\varphi| < M$, $|\nabla\varphi| < M$

故 $|u(x,t)| \le \frac{1}{4\pi a^2 t^2} \left| \int_{S_{at} \cap B_R} \psi + \varphi + at\nabla\varphi\, dS \right|$

$$\le \frac{1}{4\pi a^2 t^2} \int_{B_R} tM + M + atM\, dS$$

$$\le C_1\left(\frac{M}{t^2} + \frac{aM+1}{t}\right) \le C\frac{1}{t}$$

IV 解的渐近形态.

① $\begin{cases} u_t - a^2 u_{xx} = 0 & (t > 0,\ 0 < x < l) \\ u|_{t=0} = \varphi(x) & u(0,t) = 0 \\ u_x(l,t) + h\, u(l,t) = 0 & h > 0 \text{ 为常数} \end{cases}$

$$u(x,t) = \sum_{k=1}^{\infty} A_k e^{-a^2 \lambda_k t} \sin\sqrt{\lambda_k}\, x$$

$$A_k = \frac{1}{M_k} \int_0^l \varphi(\xi) \sin\sqrt{\lambda_k}\,\xi\, d\xi \qquad M_k = \frac{l}{2} + \frac{h}{2(h^2+\lambda_k)}$$

其中 λ_k 为 $\tan(\sqrt{x}\, l) = \frac{-\sqrt{x}}{h}$ 的解.

定理: $\varphi \in C^1$ $\varphi(0) = 0$, $\varphi'(l) + h\varphi(l) = 0$

当 $t \to +\infty$ 时 初边值问题的解指数衰减

即 $|u(x,t)| \le C e^{-a^2\lambda_1 t} \to 0$

证明: $|A_k| = \frac{1}{M_k} \left| \int_0^l \varphi(\xi) \sin\sqrt{\lambda_k}\,\xi\, d\xi \right| \le 2M \triangleq A$

由 λ_k 的估计可知 $k \to \infty$ 时 $\lambda_k = O(k^2)$

故 $\sum_{k=2}^{\infty} \frac{1}{\lambda_k - \lambda_1} < +\infty$

当 $t \geq 1$ 时 $(\lambda_k - \lambda_1) e^{-a^2(\lambda_k - \lambda_1)t} \leq B$.

$\forall x \in [0, l]$. $|u(x,t)| \leq A\left(1 + \sum_{k=2}^{\infty} e^{-a^2(\lambda_k - \lambda_1)}\right) e^{-a^2\lambda_1 t}$

$\leq A\left(1 + \sum_{k=2}^{\infty} (\lambda_k - \lambda_1) e^{-a^2(\lambda_k - \lambda_1)t} \frac{1}{\lambda_k - \lambda_1}\right) e^{-a^2\lambda_1 t}$

$\leq A\left(1 + B\sum_{k=2}^{\infty} \frac{1}{\lambda_k - \lambda_1}\right) e^{-a^2\lambda_1 t} \leq C e^{-a^2\lambda_1 t}$.

② 柯西问题 $\begin{cases} u_t - a^2 u_{xx} = 0 \\ u|_{t=0} = \varphi(x) \in L^1 \end{cases}$

定理：上述问题解 $|u(x,t)| \leq \frac{C}{t}$

证明：$|u(x,t)| \leq \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} |\varphi(\xi)| e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi$

$\leq \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} |\varphi(\xi)| d\xi \leq C \frac{1}{t}$

其中 $C = \frac{1}{2a\sqrt{\pi}} \|\varphi\|_{L^1}$.

V. 方程 $\begin{cases} u_t - a^2 u_{xx} - b u_x - c u = f(x,t) \\ u|_{t=0} = \varphi(x) \end{cases}$.

设法消去 $-bu_x$ 及 $-cu$ 两项

令 $u = e^{\lambda t + \mu x} v$. $u_t = \lambda e^{\lambda t + \mu x} v + e^{\lambda t + \mu x} v_t$

$u_x = \mu e^{\lambda t + \mu x} v + e^{\lambda t + \mu x} v_x$

$u_{xx} = \mu^2 e^{\lambda t + \mu x} v + 2\mu e^{\lambda t + \mu x} v_x + e^{\lambda t + \mu x} v_{xx}$

代入得 $\begin{cases} 2a^2\mu + b = 0 \\ \lambda - a^2\mu^2 - b\mu - c = 0 \end{cases}$

解得 $\begin{cases} \mu = -\dfrac{b}{2a^2} \\ \lambda = c - \dfrac{b^2}{4a^2} \end{cases}$

原方程化为 $\begin{cases} v_t - a^2 v_{xx} = e^{-\lambda t - \mu x} f(x,t) = g(x,t) \\ v|_{t=0} = e^{-\mu x}\varphi(x) = \psi(x) \end{cases}$

由 Fourier 变换解得

$$v(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \psi e^{-\frac{|x-\xi|^2}{4a^2 t}} d\xi$$

$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \int_{-\infty}^{+\infty} \frac{g(x,\tau)}{\sqrt{t-\tau}} e^{-\frac{|x-\xi|^2}{4a^2(t-\tau)}} d\xi\, d\tau$$

VI 三维波动方程的求解

$$\begin{cases} u_{tt} = \Delta u + 2(y-t) & x, y, z \in \mathbb{R}^3,\ t > 0. \\ u|_{t=0} = 0. & u_t|_{t=0} = x^2 + yz. \end{cases}$$

解：考虑如下的两个问题：

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u|_{t=0} = 0. \quad u_t|_{t=0} = x^2 + yz \end{cases} \qquad (\mathrm{I})$$

和 $$\begin{cases} u_{tt} - \Delta u = 2(y-t) \\ u|_{t=0} = 0. \quad u_t|_{t=0} = 0. \end{cases} \qquad (\mathrm{II})$$

由 Kirchhoff 公式.

$$u_1(x,t) = \frac{t}{4\pi} \int_0^{2\pi} \int_0^{\pi} \psi(x + \sin\theta\cos\varphi\, at,\ y + \sin\theta\sin\varphi\, at,\ z + \cos\theta\, t) \sin\theta\, d\theta\, d\varphi$$

$$= \frac{1}{3} t^2 + (x^2 + yz)t.$$

再看 (II) 的解 $u_2(x,t) = \int_0^t w(x, t; \tau)\, d\tau$

其中 $w(x,t;\tau)=2(y-\tau)(t-\tau)$

令 $\tilde{w}(x,t,\tau)=w(\vec{x},t+\tau;\tau)$

$\tilde{w}|_{t=0}=0 \quad \tilde{w}_t|_{t=0}=2(y-\tau)$

$$\tilde{w}=\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}2(y+\sin\theta\sin\varphi-\tau)\sin\theta\, d\theta\, d\varphi$$

$=2t(y-\tau)$

$\therefore w=2(t-\tau)(y-\tau)$

$$u_2=\int_0^t 2(t-\tau)(y-\tau)\,d\tau=yt^2-\frac{1}{3}t^3$$

由叠加原理知 $\vec{u}(x,t)=yt^2-\frac{1}{3}t^3+\frac{1}{3}t^3+(x^2+yz)t$

$=yt^2+(x^2+yz)t$.

第十二次习题课

I. 波方程的 Poisson 公式

$$u(x_1, x_2, t) = \frac{1}{2\pi}\int_0^t\int_0^{2\pi} \frac{\psi(x_1 + r\cos\theta, x_2 + r\sin\theta)}{\sqrt{t^2 - r^2}}\, r\, d\theta\, dr$$

$$+ \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_0^t\int_0^{2\pi} \frac{\varphi(x_1 + r\cos\theta, x_2 + r\sin\theta)}{\sqrt{t^2 - r^2}}\, r\, d\theta\, dr \right)$$

求证：
$$\begin{cases} u_{tt} = \Delta u \\ u|_{t=0} = \varphi \qquad u_t|_{t=0} = \psi \end{cases}$$

证明：定义 $\Phi = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$

$$\Psi = \frac{1}{2\pi}\int_0^{2\pi} \psi(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$$

则
$$u = \int_0^t \frac{\Psi \cdot r}{\sqrt{t^2 - r^2}}\, dr + \frac{\partial}{\partial t}\left(\int_0^t \frac{\Phi \cdot r}{\sqrt{t^2 - r^2}}\, dr \right)$$

由
$$\int_0^t \frac{\Phi \cdot r}{\sqrt{t^2 - r^2}}\, dr = -\sqrt{t^2 - r^2}\,\Phi \Big|_0^t + \int_0^t \frac{\partial \Phi}{\partial r}\sqrt{t^2 - r^2}\, dr$$

$$= t\,\varphi(x_1, x_2) + \int_0^t \frac{\partial \Phi}{\partial r}\sqrt{t^2 - r^2}\, dr$$

$$\frac{\partial}{\partial t}\left(\int_0^t \frac{\Phi \cdot r}{\sqrt{t^2 - r^2}}\, dr \right) = \varphi(x_1, x_2) + t\int_0^t \frac{\frac{\partial \Phi}{\partial r}}{\sqrt{t^2 - r^2}}\, dr$$

$u|_{t=0} = \varphi(x_1, x_2)$ 得证．

又上式
$$= \varphi(x_1, x_2) + t\left(\arcsin\frac{r}{t}\,\frac{\partial \Phi}{\partial r}\Big|_0^t - \int_0^t \arcsin\frac{r}{t}\,\frac{\partial^2 \Phi}{\partial r^2}\, dr \right)$$

$$\left(\frac{\partial}{\partial t}\right)^2 \left(\int_0^t \frac{\Phi \cdot r}{\sqrt{t^2 - r^2}}\, dr \right) = \int_0^t \frac{\frac{\partial \Phi}{\partial r}}{\sqrt{t^2 - r^2}}\, dr$$

$$+ t\left(\frac{\pi}{2}\,\frac{\partial^2 \Phi}{\partial r^2}\Big|_{r=t} - \frac{\pi}{2}\,\frac{\partial^2 \Phi}{\partial r^2}\Big|_{r=t} - \int_0^t \frac{-\frac{r}{t^2}}{\sqrt{1 - \frac{r^2}{t^2}}}\,\frac{\partial^2 \Phi}{\partial r^2}\, dr \right)$$

$$= \int_0^t \frac{\frac{\partial \Phi}{\partial r}}{\sqrt{t^2 - r^2}}\, dr + \int_0^t \frac{r}{\sqrt{t^2 - r^2}}\,\frac{\partial^2 \Phi}{\partial r^2}\, dr$$

由上式得 $u_t\big|_{t=0} = \varphi(x_1,x_2) + 0 = \varphi(x_1,x_2)$

又 $u_{tt} = \int_0^t \frac{\Phi_r}{\sqrt{t^2-r^2}}\,dr + \int_0^t \frac{r\Phi_{rr}}{\sqrt{t^2-r^2}}\,dr$

$+\frac{\partial}{\partial t}\left(\int_0^t \frac{\Psi_r}{\sqrt{t^2-r^2}}\,dr + \int_0^t \frac{r\Psi_{rr}}{\sqrt{t^2-r^2}}\,dr\right)$

$\Delta u = \int_0^t \frac{\Delta\Phi\cdot r}{\sqrt{t^2-r^2}}\,dr + \frac{\partial}{\partial t}\left(\int_0^t \frac{\Delta\Psi\cdot r}{\sqrt{t^2-r^2}}\,dr\right)$

只要证 $\int_0^t \frac{\Phi_r + r\Phi_{rr}}{\sqrt{t^2-r^2}}\,dr = \int_0^t \frac{\Delta\Phi\cdot r}{\sqrt{t^2-r^2}}\,dr$ 即可.

注意到 $\Delta\Phi = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta}$

且 Φ 与 θ 无关即得证.

II. Fourier 变换与调和分析初步

例1: 求出 $\widehat{e^{-\pi x^2}}$

解: $\widehat{e^{-\pi x^2}} = \int_{\mathbb{R}} e^{-\pi x^2 - 2\pi i x\xi}\,dx$

$= \int_{\mathbb{R}} e^{-\pi x^2}\cos 2\pi x\xi\,dx - i\int_{\mathbb{R}} e^{-\pi x^2}\sin 2\pi x\xi\,dx$

$= \int_{\mathbb{R}} e^{-\pi x^2}\cos 2\pi x\xi\,dx \triangleq I(\xi)$.

$I'(\xi) = -\int_{\mathbb{R}} 2\pi x\, e^{-\pi x^2}\sin 2\pi x\xi\,dx$

$= e^{-\pi x^2}\sin 2\pi x\xi\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} 2\pi\xi\, e^{-\pi x^2}\cos 2\pi x\xi\,dx$

$\Rightarrow I'(\xi) = -2\pi\xi\, I(\xi)$

$\Rightarrow I(\xi) = C e^{-\pi\xi^2}$. 由 $I(0)=1$ 知 $C=1$

从而有 $\widehat{e^{-\pi x^2}} = e^{-\pi\xi^2}$

例2：A为$n\times n$正定对称矩阵. 求 $\widehat{e^{-\pi x^TAx}}$

解：设 $A=P^TP$. 记 $y=Px$, $\eta^T=\xi^TP^{-1}$.

$$\int_{\mathbb{R}^n} e^{-\pi x^TAx-2\pi i\xi^Tx}\,dx$$

$$=\int_{\mathbb{R}^n} e^{-\pi y^Ty-2\pi i\eta^Ty}\frac{1}{|P|}\,dy$$

$$=\frac{1}{|A|^{\frac12}}\prod_{i=1}^n\int_{\mathbb{R}} e^{-\pi y_i^2-2\pi i\eta_iy_i}\,dy_i$$

$$=\frac{1}{|A|^{\frac12}}\prod_{i=1}^n e^{-\pi\eta_i^2}=\frac{1}{|A|^{\frac12}}e^{-\pi(\xi^TP^{-1})(P^{T^{-1}}\xi)}$$

$$=\frac{1}{|A|^{\frac12}}e^{-\pi\xi^TA^{-1}\xi}$$

定义：设 $f\in C^\infty(\mathbb{R})$. 若$\forall\alpha,\beta\in\mathbb{N}$. 有常数 $C_{\alpha,\beta}$.

s.t. $\left|x^\alpha\left(\frac{d}{dx}\right)^\beta f\right|\leq C_{\alpha,\beta}$. 则称 $f\in S$.

S称为Schwarz空间或速降空间.

易见 $C_c^\infty\subsetneq S\subsetneq C_0^\infty$.

例子：$e^{-\pi x^2}$, $e^{-\pi x^TAx}\in S$.

但 $\frac{1}{1+x}$, $\frac{1}{1+x^2}\notin S$.

由Uryshon引理知 C_c^∞ 在 L^2 中稠密

从而 S 在 L^2 中也稠密. 下面定义 L^2 上的Fourier变换

性质①：设 $f,g\in S$. 则有

$$\int_{\mathbb{R}}\hat f(x)g(x)\,dx=\int_{\mathbb{R}}f(x)\hat g(x)\,dx$$

证明：易见 左 $= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)\, e^{-2\pi i x y} g(x)\, dy\, dx$

$= \int_{\mathbb{R}}\int_{\mathbb{R}} g(y)\, e^{-2\pi i x y} f(x)\, dx\, dy =$ 右

性质② $\overline{\hat{f}}(\xi) = \check{\overline{f}}(\xi)$.

证明：$\overline{\hat{f}}(\xi) = \overline{\int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx} = \int_{\mathbb{R}} \overline{f(x)}\, e^{2\pi i x \xi} dx$

$= \check{\overline{f}}(\xi)$. 从而结论得证.

在①中令 $g = \overline{\hat{f}} = \check{\overline{f}}$ 得 $\int_{\mathbb{R}} |\hat{f}|^2 dx = \int_{\mathbb{R}} |f|^2 dx$.

$\Rightarrow \|f\|_{L^2} = \|\hat{f}\|_{L^2}$. 此式称为 Plancherel 等式

下面说明如何定义 L^2 上的 Fourier 变换

设 $f \in L^2$. 则 $\exists f_k \in S$. $f_k \to f$ in L^2.

即 $\|f_k - f\|_{L^2} \to 0 \quad \Rightarrow \|f_m - f_n\|_{L^2} \to 0$

$\Rightarrow \|\hat{f}_m - \hat{f}_n\|_{L^2} \to 0 \quad \Rightarrow \hat{f}_n \to g$ in L^2.

定义 g 为 f 的 Fourier 变换. 记 $g = \hat{f}$.

易知 g 的唯一性. 这就定义了 L^2 上的 Fourier 变换.

例3：求证 $\|f * g\|_{L^2}^2 \le \|f * f\|_{L^2} \|g * g\|_{L^2}$.

证：只要证 $\|\hat{f}\cdot\hat{g}\|_{L^2}^2 \le \|\hat{f}\cdot\hat{f}\|_{L^2} \|\hat{g}\cdot\hat{g}\|_{L^2}$

即 $\int |\hat{f}|^2 |\hat{g}|^2 \le \left(\int |\hat{f}|^4\right)^{\frac{1}{2}} \left(\int |\hat{g}|^4\right)^{\frac{1}{2}}$

即 $\left(\int |\hat{f}|^2 |\hat{g}|^2\right)^2 \le \left(\int |\hat{f}|^4\right)\left(\int |\hat{g}|^4\right)$

此即 Cauchy – Schwarz 不等式.

例4：Heinserberg 不确定性原理

已知 $\varphi \in S(\mathbb{R})$，且 $\int_{-\infty}^{+\infty} |\varphi(x)|^2 dx = 1$

求证 $\int_{-\infty}^{+\infty} x^2|\varphi(x)|^2 dx \int_{-\infty}^{+\infty} \xi^2 |\hat{\varphi}(\xi)|^2 d\xi \geq \frac{1}{16\pi^2}$

证：$1 = \int_{-\infty}^{+\infty} |\varphi(x)|^2 dx = x\varphi^2(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} 2x\varphi(x)\varphi'(x) dx$

$\leq 2\int_{-\infty}^{+\infty} x|\varphi(x)||\varphi'(x)| dx \leq 2\left(\int_{-\infty}^{+\infty} x^2|\varphi(x)|^2 dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}|\varphi'(x)|^2 dx\right)^{\frac{1}{2}}$

而 $\|\varphi'\|_{L^2} = \|\hat{\varphi'}\|_{L^2} = \|2\pi i\xi\hat{\varphi}(\xi)\|_{L^2}$

$= 2\pi\left(\int_{-\infty}^{+\infty} \xi^2|\hat{\varphi}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$ 代入得

$1 \leq 4\pi\left(\int_{-\infty}^{+\infty} x^2|\varphi(x)|^2 dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty} \xi^2|\hat{\varphi}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$

$\Rightarrow \int_{-\infty}^{+\infty} x^2|\varphi(x)|^2 dx \int_{-\infty}^{+\infty} \xi^2|\hat{\varphi}(\xi)|^2 d\xi \geq \frac{1}{16\pi^2}$

易知将原题中的不等式改为

$$\int_{-\infty}^{+\infty} (x-x_0)^2|\varphi(x)|^2 dx \int_{-\infty}^{+\infty} (\xi-\xi_0)^2|\hat{\varphi}(\xi)|^2 d\xi \geq \frac{1}{16\pi^2}$$

其中 x_0，ξ_0 为任意常数，则依然成立

Ⅲ 边界梯度估计（Bernstein 技巧）

例1：设 $u \in C^\infty(\overline{B_1(0)})$ $\Delta u = 0$ in $B_1(0)$

求证：$\sup\limits_{B_1(0)} |\nabla u| \leq \sup\limits_{\partial B_1(0)} |\nabla u|$

解：$\Delta(|\nabla u|^2) = (u_j^2)_{ii} = (2u_j u_{ji})_i$

$= 2u_{ij}^2 + 2u_j u_{iij} = 2u_{ij}^2 \geq 0$

故 $\sup\limits_{B_1(0)} |\nabla u|^2 \leq \sup\limits_{\partial B_1(0)} |\nabla u|^2 \Rightarrow \sup\limits_{B_1(0)} |\nabla u| \leq \sup\limits_{\partial B_1(0)} |\nabla u|$

例2： 设 $u \in C^\infty(\overline{B_1(0)})$ $\Delta u + u = 0$

求证： $\sup\limits_{B_1(0)} |\nabla u| \le \sup\limits_{\partial B_1(0)} |\nabla u| + C \sup\limits_{B_1(0)} |u|$

解： $\Delta(|\nabla u|^2) = 2u_{ij}^2 + 2u_i(\Delta u)_i$

$= 2u_{ij}^2 - 2u_i^2 = 2|D^2u|^2 - 2|\nabla u|^2$.

$\Delta(u^2) = (2uu_i)_i = 2u_i^2 + 2u\Delta u = 2|\nabla u|^2 - 2u^2$

$\Delta\left(\frac{|x|^2}{n} \max\limits_{B_1(0)} |u|^2\right) = 2\max\limits_{B_1(0)} |u|^2$

从而 $\Delta\left(|\nabla u|^2 + u^2 + \frac{|x|^2}{n} \max\limits_{B_1(0)} |u|^2\right) \ge 0$

故 $\max\limits_{\overline{B(0)}} |\nabla u|^2 \le \max\limits_{\overline{B_1(0)}} \left(|\nabla u|^2 + u^2 + \frac{|x|^2}{n} \max\limits_{B_1(0)} |u|^2\right)$

$\le \max\limits_{\partial B_1(0)} |\nabla u|^2 + 2\max\limits_{\overline{B_1(0)}} |u|^2$

$\Rightarrow \max\limits_{\overline{B_1(0)}} |\nabla u| \le \max\limits_{\partial B_1(0)} |\nabla u| + 2\max\limits_{\overline{B_1(0)}} |u|$

例3： $\Omega \subset \mathbb{R}^n$ 有界. $\Omega \subset \{a < x_1 < b\}$.

$$\begin{cases} Lu = a^{ij}u_{ij} = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

则 $\max\limits_{\Omega} |u| \le C\left(\max\limits_{\partial\Omega} |\varphi| + \max\limits_{\Omega} |f|\right)$

解： $v(x) \triangleq \Phi + \left(e^{\mu(b-a)} - e^{\mu(x_1-a)}\right)F$

$a^{ij}v_{ij} = a^{ij}\left(-\mu^2 e^{\mu(x_1-a)} \delta_{1i}\delta_{1j}\right)F$

$= -a^{11}\mu^2 e^{\mu(x_1-a)} F \le -a^{11}\mu^2 F \le -F$

（取 μ 充分大即有上式） 即 $\begin{cases} Lv \le \pm Lu & \text{in } \Omega \\ v \ge \pm u & \text{on } \partial\Omega \end{cases}$

从而 $|u| \le v \le \max_{\partial\Omega}|\varphi| + C\max_{\Omega}|f|$.

事实上，比较定理对于非线性椭圆方程也是成立

例如 Monge-Ampère 方程有如下的比较原理

设 u, v 为凸函数，且 $\begin{cases} \det u_{ij} \ge \det v_{ij} & \text{in } \Omega \\ u \le v & \text{on } \partial\Omega \end{cases}$.

则有 $u \le v$ in Ω 成立.

Ⅳ 微分的局部估计

设 $\Delta u = 0$ in $B_r(x_0)$，则有如下估计式

$$|D^k u(x_0)| \le \frac{C_k}{r^k}\|u\|_{L^\infty(B_r(x_0))}$$

证明：只要证 $k=1$ 的情形即可

$$u_i(x_0) = \fint_{B_r(x_0)} u_i(x)\,dx = \frac{1}{\omega_n r^n}\int_{B_r(x_0)} u_i(x)\,dx$$

$$= \frac{1}{\omega_n r^n}\int_{\partial B_r(x_0)} u\cdot\nu_i\,dx$$

$$|u_i(x_0)| \le \frac{1}{\omega_n r^n}\int_{\partial B_r(x_0)}\|u\|_{L^\infty}dx \le \frac{n}{r}\|u\|_{L^\infty(B_r(x_0))}$$

$$\Rightarrow |Du(x_0)| \le \frac{C}{r}\|u\|_{L^\infty(B_r(x_0))}$$

则在 $k=2$ 时，用上式 $|D^2u(x_0)| \le \frac{C_1}{\frac{r}{2}}\|u\|_{L^\infty(B_{\frac{r}{2}}(x_0))}$

$$\le \frac{C_1C_2}{\frac{r}{2}\frac{r}{2}}\|u\|_{L^\infty(B_r(x_0))} \le \frac{C}{r^2}\|u\|_{L^\infty(B_r(x_0))}$$

同理可知 $|D^k u(x_0)| \le \frac{C_k}{r^k}\|u\|_{L^\infty(B_r(x_0))}$

$$\Rightarrow |D^k u(x_0)| \lesssim \frac{1}{r^k}\|u\|_{L^\infty(B_r(x_0))}$$

V 注释家的 Schauder 内估计

定理：设 $\Delta u = f$ in $B_1(0)$. 记 "$\lesssim$" 为 "$\le C$"

设 $[f]_{C^\alpha} \triangleq \sup\limits_{x\neq y} \dfrac{|f(x)-f(y)|}{|x-y|^\alpha}$　　$\omega(r) = \sup\limits_{|x-y|<r} |f(x)-f(y)|$

则有 $[D^2u]_{C^\alpha(B_{\frac{1}{2}})} \lesssim \|D^2u\|_{L^\infty} + \|u\|_{L^\infty} + [f]_{C^\alpha}$.

证明：记 $B_k(x_0) \triangleq B_{2^{-k}}(x_0)$ $(k\in N)$.

取 $\begin{cases} \Delta u_k = f(x_0) & \text{in } B_k(x_0) \\ u_k = u & \text{on } \partial B_k(x_0) \end{cases}$　$\begin{cases} \Delta v_k = f(y_0) & \text{in } B_k(y_0) \\ v_k = u & \text{on } \partial B_k(y_0) \end{cases}$

考虑有如下关系 $\begin{cases} \Delta(u_k - u) = f(x_0) - f & \text{in } B_k(x_0) \\ u_k - u = 0 & \text{on } \partial B_k(x_0) \end{cases}$

定义 $\Phi \triangleq \frac{1}{2n} w(2^{-k})\left(2^{-2k} - |x-x_0|^2\right)$

则 $\begin{cases} \Delta\Phi = -w(2^{-k}) \le \pm\Delta(u_k - u) & \text{in } B_k(x_0) \\ \Phi = 0 = \pm(u_k - u) & \text{on } \partial B_k(x_0) \end{cases}$

从而 $\|u_k - u\|_{L^\infty(B_k(x_0))} \le \|\Phi\|_{L^\infty} \le \frac{1}{2n} w(2^{-k})\, 2^{-2k}$.

故 $\|u_{k+1} - u\|_{L^\infty(B_{k+1}(x_0))} \le \frac{1}{2n} w(2^{-k-1})\, 2^{-2k-2}$.

$\Rightarrow \|u_{k+1} - u_k\|_{L^\infty} \le \|u_k - u\|_{L^\infty} + \|u - u_{k+1}\|_{L^\infty}$

$\le 2\times\frac{1}{2n} w(2^{-k})\, 2^{-2k} \lesssim w(2^{-k})\, 2^{-2k}$.

但 $\Delta(u_k - u_{k+1}) = f(x_0) - f(x_0) = 0$.

故 $|D^2u_k(x_0) - D^2u_{k+1}(x_0)| \lesssim 2^{2k}\|u_{k+1} - u_k\|_{L^\infty} \lesssim w(2^{-k})$

又 $u(x) = u(x_0) + Du(x_0)(x-x_0) + \frac{1}{2}(x-x_0)^T D^2u(x_0)(x-x_0) + o(|x-x_0|^2)$

定义 $P \triangleq u(x_0) + Du(x_0)(x-x_0) + \frac{1}{2}(x-x_0)^T D^2u(x_0)(x-x_0)$

则 $\Delta(u_k - P) = 0$

又 $\|u_k - P\|_{L^\infty} \leq \|u_k - u\|_{L^\infty} + \|u - P\|_{L^\infty}$

$\lesssim w(2^{-k})2^{-2k} + o(2^{-2k}) = o(2^{-2k})$

$\Rightarrow |D^2u_k(x_0) - D^2P(x_0)| \lesssim 2^{2k} o(2^{-2k}) = o(1)$.

即 $\lim\limits_{k\to\infty} D^2u_k(x_0) = D^2P(x_0) = D^2u(x_0)$

回到原理. 若 $|x_0 - y_0| \geq \frac{1}{8}$. 则有

$$\frac{|D^2u(x_0) - D^2u(y_0)|}{|x_0 - y_0|^\alpha} \leq \frac{2\|D^2u\|_{L^\infty}}{(\frac{1}{8})^\alpha} \lesssim \|D^2u\|_{L^\infty}.$$

当 $0 < |x_0 - y_0| < \frac{1}{8}$ 时. $\exists k \in N$.

s.t. $2^{-(k+3)} \leq |x_0 - y_0| \leq 2^{-(k+2)}$

则 $|D^2u(x_0) - D^2u(y_0)| \leq |D^2u(x_0) - D^2u_k(x_0)|$

$+ |D^2u_k(x_0) - D^2v_k(y_0)| + |D^2v_k(y_0) - D^2u(y_0)|$

$\triangleq I_1 + I_2 + I_3$.

考虑 I_1: $\because (D^2u - D^2u_k)(x_0) = \sum\limits_{m=k}^{\infty}(D^2u_{m+1}(x_0) - D^2u_m(x_0))$

$\therefore |I_1| \leq \sum\limits_{m=k}^{\infty} |D^2u_{m+1}(x_0) - D^2u_m(x_0)|$

$\leq \sum\limits_{m=k}^{\infty} w(2^{-m}) \lesssim \sum\limits_{m=k}^{\infty} [f]_{C^\alpha} 2^{-m\alpha}$

$\lesssim [f]_{C^\alpha} 2^{-k\alpha} \lesssim [f]_{C^\alpha} |x_0 - y_0|^\alpha$

同理有 $I_3 \lesssim [f]_{C^\alpha} |x_0 - y_0|^\alpha$.

再考虑 I_2: $I_2 = |D^2u_k(x_0) - D^2u_k(y_0)| + |D^2u_k(y_0) - D^2v_k(y_0)|$

$\triangleq I_{21} + I_{22}$ 先考虑 I_{21}

定义 $h_j = u_j - u_{j-1}$. 则 $D^2 u_k = D^2 u_1 + \sum_{j=2}^{k} D^2 h_j$.

$$|D^2 u_k(x_0) - D^2 u_k(y_0)| \le |D^2 u_1(x_0) - D^2 u_1(y_0)|$$

$$+ \sum_{j=2}^{k} |D^2 h_j(x_0) - D^2 h_j(y_0)| \triangleq I_{211} + I_{212}.$$

考虑 I_{211}： 令 $\Psi = u_1 + \frac{f(x_0)}{2n}\left(\frac{1}{4} - |x-x_0|^2\right)$

则 $\begin{cases} \Delta\Psi = \Delta u_1 - f(x_0) = 0 & \text{in } B_{\frac{1}{2}}(x_0) \\ \Psi = u & \text{on } \partial B_{\frac{1}{2}}(x_0) \end{cases}$.

$$I_{211} = |D^2 u_1(x_0) - D^2 u_1(y_0)| \le |x_0 - y_0|\,|D^3 u_1(\tilde{x})|$$

$$= |x_0 - y_0|\,|D^3 \Psi(\tilde{x})| \lesssim |x_0 - y_0|\,\|\Psi\|_{L^\infty(B_{\frac{1}{2}}(x_0))}$$

$$\le |x_0 - y_0|\,\|u\|_{L^\infty(\partial B_{\frac{1}{2}}(x_0))} \le |x_0 - y_0|^\alpha \|u\|_{L^\infty}$$

再考虑 I_{212}： $|D^2 h_j(x_0) - D^2 h_j(y_0)| \le |x_0 - y_0|\,|D^3 h_j(\tilde{x})|$

$$\lesssim |x_0 - y_0|\, 2^{3j} \|h_j\|_{L^\infty(B_j(x_0))} \lesssim |x_0 - y_0|\, 2^{j} w(2^{1-j})$$

$$\lesssim |x_0 - y_0|\, 2^{j} [f]_{C^\alpha} 2^{(1-j)\alpha}$$

$$I_{212} \lesssim \sum_{j=2}^{k} |x_0 - y_0| [f]_{C^\alpha} 2^{j(1-\alpha)}$$

$$\lesssim [f]_{C^\alpha} |x_0 - y_0|\, 2^{k(1-\alpha)} \lesssim [f]_{C^\alpha} |x_0 - y_0|^\alpha.$$

从而有 $I_{21} = I_{211} + I_{212} \lesssim (\|u\|_{L^\infty} + [f]_{C^\alpha})\,|x_0 - y_0|^\alpha$.

最后考虑 I_{22}： 定义 $\Theta = u_k - v_k - \frac{f(x_0) - f(y_0)}{2n}|x - y_0|^2$.

则 $\Delta\Theta = 0$. $D^2\Theta = D^2 u_k - D^2 v_k - \frac{I}{n}(f(x_0) - f(y_0))$

$$|D^2 u_k(y_0) - D^2 v_k(y_0)| \le |D^2\Theta| + C\,|f(x_0) - f(y_0)|$$

$$\lesssim 2^{2k} \|\Theta\|_{L^\infty(B_k(y_0))} + W(2^{-k})$$

又 $\|\ominus\|_{L^\infty(B_k(y_0))} \leq \|u_k - v_k\|_{L^\infty(B_k(y_0))} + \frac{w(2^{-k})}{2n} 2^{-2k}$.

故有 $I_{22} \lesssim 2^{2k}\|u_k - v_k\|_{L^\infty} + w(2^{-k})$

$\leq 2^{2k}\left(\|u_k - u\|_{L^\infty} + \|v_k - u\|_{L^\infty}\right) + w(2^{-k})$

$\lesssim 2^{2k}\left(w(2^{-k}) + w(2^{-k})\right)2^{-2k} + w(2^{-k}) \lesssim w(2^{-k})$.

$\lesssim [f]_{C^\alpha} 2^{-k\alpha} \lesssim [f]_{C^\alpha} |x_0 - y_0|^\alpha$.

综上所述. 我们得到了 $\forall x_0, y_0 \in B_{\frac{1}{2}}(0)$

$|D^2u(x_0) - D^2u(y_0)| \lesssim |x_0 - y_0|^\alpha \left(\|D^2u\|_{L^\infty} + \|u\|_{L^\infty} + [f]_{C^\alpha}\right)$

从而 $[D^2u]_{C^\alpha} \lesssim \|D^2u\|_{L^\infty} + \|u\|_{L^\infty} + [f]_{C^\alpha}$.

又由插值理论知 $\|D^2u\|_{L^\infty} \leq \varepsilon [D^2u]_{C^\alpha} + C_\varepsilon \|u\|_{L^\infty}$

从而有 $[D^2u]_{C^\alpha} \lesssim \|u\|_{L^\infty} + [f]_{C^\alpha}$.

再由插值理论可知

$\|u\|_{C^{2,\alpha}} \lesssim \|u\|_{L^\infty} + \|f\|_{C^\alpha}$. 此即 Schauder 内估计

其中 $\|u\|_{C^{2,\alpha}} \triangleq \|u\|_{L^\infty} + \|Du\|_{L^\infty} + \|D^2u\|_{L^\infty} + [D^2u]_{C^\alpha}$.

$\|f\|_{C^\alpha} \triangleq \|f\|_{L^\infty} + [f]_{C^\alpha}$

之后的课程会将其推广到一般区域 Ω 上的

一般线性椭圆算子情形:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Schauder 估计可用于解决上述方程解的存在性问题

第十三次 习题课

I. Green 函数对称性

证明：$\forall x_1, x_2 \in \Omega$，$(x_1 \neq x_2)$ 令 r 充分小 s.t.

$B_r(x_1) \subset \Omega$，$B_r(x_2) \subset \Omega$．$B_r(x_1) \cap B_r(x_2) = \phi$

设 $G_1(y) = G(x_1, y)$，$G_2(y) = G(x_2, y)$

$\Gamma_1(y) = \Gamma(x_1 - y)$，$\Gamma_2(y) = \Gamma(x_2 - y)$．由第二Green公式

$$0 = \int_{\Omega \setminus (B_r(x_1) \cup B_r(x_2))} (G_1 \Delta G_2 - G_2 \Delta G_1)\, dy$$

$$= \int_{\partial\Omega} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS_y$$

$$+ \int_{\partial B_r(x_1)} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS_y + \int_{\partial B_r(x_2)} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS_y$$

$\vec{n}$ 为 $\partial(\Omega \setminus (B_r(x_1) \cup B_r(x_2)))$ 的单位外法向量．

又由 $G_1(y) = 0$，$G_2(y) = 0$ on $\partial\Omega$．故有

$$\int_{\partial B_r(x_1)} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS_y + \int_{\partial B_r(x_2)} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS_y = 0$$

又 $\because$ $G_1 - \Gamma_1$ 在 Ω 内调和，G_2 在 $\Omega \setminus B_r(x_2)$ 上调和．

故有 $\displaystyle\int_{\partial B_r(x_1)} \left((G_1 - \Gamma_1)\frac{\partial G_2}{\partial n} - G_2 \frac{\partial (G_1 - \Gamma_1)}{\partial n}\right) dS_y$

$$= \int_{B_r(x_1)} \left((G_1 - \Gamma_1)\Delta G_2 - G_2 \Delta (G_1 - \Gamma_1)\right) dy = 0$$

同理有 $\displaystyle\int_{\partial B_r(x_2)} \left((G_2 - \Gamma_2)\frac{\partial G_1}{\partial n} - G_1 \frac{\partial (G_2 - \Gamma_2)}{\partial n}\right) dS_y = 0$

因为 $\displaystyle\left|\int_{\partial B_r(x_1)} \Gamma_1 \frac{\partial G_2}{\partial n} dS_y\right| = \left|\int_{\partial B_r(x_1)} \frac{r^{2-n}}{n(2-n)\omega_n} \frac{\partial G_2}{\partial n} dS_y\right|$

$$\le \frac{r}{n-2}\max_{\partial B_r(x_1)}\left|\frac{\partial G_2}{\partial n}\right| \to 0 \quad (r\to 0).$$

同理有 $\left|\int_{\partial B_r(x_2)} \Gamma_2 \frac{\partial G_1}{\partial n} dS_y\right| \to 0 \quad (r \to 0)$

又因为 $\int_{\partial B_r(x_1)} G_2 \frac{\partial \Gamma_1}{\partial n} dS_y = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x_1)} G_2\, dS_y \to G_2(x_1)$

$\int_{\partial B_r(x_2)} G_1 \frac{\partial \Gamma_2}{\partial n} dS_y = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x_2)} G_1\, dS_y \to G_1(x_2) \quad (r\to 0).$

综上令 $r\to 0$ 有 $G_1(x_2) = G_2(x_1)$.

即有 $G(x_1, x_2) = G(x_2, x_1)$ 得证.

Ⅱ. Hadamard 三圆定理.

定理. 设 D 为 $\mathbb{R}^2$ 中以原点为中心的环形区域, 大小圆半径分别为 R_2, R_1. $\Delta u \ge 0$ in D.

记 $M(r) = \max\limits_{x^2+y^2=r^2} u(x,y)$ $\quad R_1 < r_1 < r < r_2 < R_2$.

其中 $r = \sqrt{x^2+y^2}$. 则有

$$M(r) \le \frac{M(r_1)\ln\frac{r_2}{r} + M(r_2)\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}$$

证明: 令 $\varphi(r) = a + b\ln r \quad (r \ne 0)$.

由 $\varphi(r_1) = M(r_1)$ $\quad \varphi(r_2) = M(r_2)$ 可确定 a, b.

代入有 $\varphi(r) = \dfrac{M(r_1)\ln\frac{r_2}{r} + M(r_2)\ln\frac{r}{r_1}}{\ln\left(\frac{r_2}{r_1}\right)}$

设 $V(x,y) = u(x,y) - \varphi(\sqrt{x^2+y^2})$

则 $\begin{cases} \Delta V \ge 0 & \text{当 } r_1 < r < r_2 \text{ 时} \\ V \le 0 & \text{当 } r = r_1 \text{ 和 } r = r_2 \text{ 时} \end{cases}$

由极值原理知 $V \le 0$ in $r_1 < r < r_2$

因此 $u(x,y) \le \varphi(r) \Rightarrow M(r) \le \varphi(r)$

即 $$M(r) \le \frac{M(r_1)\ln\frac{r_2}{r} + M(r_2)\ln\frac{r}{r_1}}{\ln\left(\frac{r_2}{r_1}\right)}$$

Ⅲ 第三边值问题解的唯一性.

定理1: $$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \sigma u = 0 & \text{on } \partial\Omega \end{cases} \quad (\sigma > 0)$$

其中 Ω 满足内球条件. 则 $u \equiv 0$

证明: 若 u 不恒为 0. 则必有

(1) $\alpha \triangleq \max\limits_{x\in\bar\Omega} u = \max\limits_{x\in\partial\Omega} u > 0$ 或者

(2) $\beta \triangleq \min\limits_{x\in\bar\Omega} u = \min\limits_{x\in\partial\Omega} u < 0$.

若 (1) 成立. 设 $x_0 \in \partial\Omega$. $u(x_0) = \alpha > 0$. 则由

Hopf 引理知 $\left.\frac{\partial u}{\partial n}\right|_{x_0} > 0 \Rightarrow \left.\frac{\partial u}{\partial n}\right|_{x_0} + \sigma\alpha > 0$. 矛盾

若 (2) 成立. 设 $y_0 \in \partial\Omega$. $u(y_0) = \beta < 0$. 则由

Hopf 引理知 $\left.\frac{\partial u}{\partial n}\right|_{x_0} < 0 \Rightarrow \left.\frac{\partial u}{\partial n}\right|_{x_0} + \sigma\beta < 0$ 矛盾

故知 $u \equiv 0$ 从而可得唯一性定理如下

定理2: $$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \sigma u = \varphi & \text{on } \partial\Omega \end{cases}$$

的解若存在则一定唯一.

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《数学物理方程》谷超豪 李大潜

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