EX 3, 2020.3.24

1. Let $\{X_n\}$ be a sequence of random variables such that $EX_n = m$ and $\operatorname{Var}(X_n) = \sigma_n^2 > 0$ for all n, where $\sigma_n^2 \to 0$ as $n \to \infty$. Define

$$Z_n = \sigma_n^{-1}(X_n - m),$$

and let f be a function with non-zero derivative $f'(m)$ at m.
(a) Show that $Z_n = O_p(1)$ and $X_n = m + o_p(1)$.
(b) If $Y_n = [f(X_n) - f(m)] / [\sigma_n f'(m)]$, show that $Y_n - Z_n = o_p(1)$.
(c) Show that if Z_n converges in probability or in distribution then so does Y_n.
(d) If S_n is binomially distributed with parameters n and p, and $f'(p) \neq 0$, use the preceding results to determine the asymptotic distribution of $f(S_n/n)$.

2. Suppose that X_n is AN (μ, σ_n^2) where $\sigma_n^2 \to 0$. Show that $X_n \xrightarrow{P} \mu$.

If $\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1)$, denote $X_n \sim \text{AN}(\mu, \sigma_n^2)$

3. If X_n is $AN(\mu_n, \sigma_n^2)$, show that
(a) X_n is AN $(\tilde{\mu}_n, \tilde{\sigma}_n^2)$ if and only if $\tilde{\sigma}_n / \sigma_n \to 1$ and $(\tilde{\mu}_n - \mu_n)/\sigma_n \to 0$, and
(b) $a_n X_n + b_n$ is AN (μ_n, σ_n^2) if and only if $a_n \to 1$ and $(\mu_n(a_n - 1) + b_n)/\sigma_n \to 0$.
(c) If X_n is AN$(n, 2n)$, show that $(1 - n^{-1}) X_n$ is AN$(n, 2n)$ but that $(1 - n^{-1/2}) X_n$ is not AN$(n, 2n)$.

4. If $X_1, X_2, \ldots,$ are iid normal random variables with mean μ and variance σ^2, find the asymptotic distributions of $\bar{X}_n^2 = \left(n^{-1}\sum_{j=1}^n X_j\right)^2$
(a) when $\mu \neq 0$, and
(b) when $\mu = 0$.

5. $E\left(X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}\right) = \frac{1}{\mathrm{j}^{k_1+k_2+\cdots+k_n}} \frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial\omega_1^{k_1}\partial\omega_2^{k_2}\cdots\partial\omega_n^{k_n}} \phi_{\boldsymbol{X}}(\omega_1, \omega_2, \cdots, \omega_n)\Big|_{\omega_1=\omega_2=\cdots=\omega_n=0}$

$$E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

$$E(X_1X_2X_3) = E(X_1X_2)E(X_3) + E(X_1X_3)E(X_2) + E(X_1)E(X_2X_3)$$

$$EX_1^2X_2^2 = EX_1^2EX_2^2 + 2(EX_1X_2)^2$$