EX 7 简单的答案解析

2021 年 5 月 17 日

1 判断是否为因果和可逆的

(a) 模型是因果可逆的。

$A(z) = 1 + 0.2z - 0.48z^2, B(z) = 1$

$A(z) = 0 \Rightarrow z_1 = -1.25, z_2 = 5/3$

(b) 模型不是因果但是可逆的。

$A(z) = 1 + 1.9z + 0.88z^2, B(z) = 1 + 0.2z + 0.7z^2$

$A(z) = 0 \Rightarrow z_1 = -\frac{10}{11}, z_2 = -1.25, |z_1| < 1$

$B(z) = 1 + 0.2z + 0.7z^2 > 0$ 恒成立。

(c) 模型是因果但是不可逆的。

$A(z) = 1 + 0.6z^2, B(z) = 1 + 1.2z$

$A(z) = 1 + 0.6z^2 > 0$ 恒成立。

$B(z) = 0 \Rightarrow z = -\frac{5}{6}, |z| < 1$

(d) 模型是因果可逆的。

$A(z) = 1 + 1.8z + 0.81z^2, B(z) = 1$

$A(z) = 0 \Rightarrow z_1 = z_2 = -\frac{10}{9}, |z| > 1$

(e) 模型不是因果但是可逆的。

$A(z) = 1 + 0.6z, B(z) = 1 - 0.4z + 0.04z^2$

$A(z) = 0 \Rightarrow z = -\frac{5}{8}, |z| < 1$

$B(z) = 1 - 0.4z + 0.04 = 0 \Rightarrow z_1 = z_2 = -5, |z| > 1$

2 有相同的自相关系数

对于 X_t,

$$\gamma_0 = \sigma^2 \textstyle\sum_{j=0}^{1} b_j^2 = \sigma^2\left(b_0^2 + b_1^2\right) = \sigma^2\left(1 + \theta^2\right)$$
$$\gamma_1 = \sigma^2 b_0 b_1 = \theta\sigma^2$$
$$\gamma_k = 0, |k| > 1$$

对于 Y_t,

$$\gamma_0' = \sigma^2\theta^2 \textstyle\sum_{j=0}^{1} b_j'^2 = \sigma^2\theta^2\left(1 + \frac{1}{\theta^2}\right) = \sigma^2\left(1 + \theta^2\right)$$
$$\gamma_1' = \sigma^2\theta^2 b_0' b_1' = \theta\sigma^2$$
$$\gamma_k' = 0, |k| > 1$$

故 X_t, Y_t 有相同的自相关系数。

3 (1)

$$X_t = \phi_0 + \phi_1 X_{t-1} + \varepsilon_t - \theta_1\varepsilon_{t-1} \Rightarrow X_t - \frac{\phi_0}{1-\phi_1} = \phi_1\left(X_{t-1} - \frac{\phi_0}{1-\phi_1}\right) + \varepsilon_t - \theta_1\varepsilon_{t-1}$$

$$Y_t = X_t - \frac{\phi_0}{1-\phi_1}, Y_t = \phi_1 Y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$A(z) = 1 - \phi_1 z \neq 0, |z| \leq 1; B(z) = 1 - \theta_1 z \neq 0, |z| \leq 1$$

$$A(L) Y_t = B(L) \varepsilon_t \Rightarrow Y_t = \frac{B(L)}{A(L)} \varepsilon_t, \varepsilon_t = \frac{A(L)}{B(L)} Y_t$$

$$\frac{B(z)}{A(z)} = \frac{1-\theta_1 z}{1-\phi_1 z} = (1-\theta_1 z) \sum_{j=0}^{\infty} (\phi_1 z)^j = 1 + \sum_{j=1}^{\infty} (\phi_1 - \theta_1) \phi_1^{j-1} z^j$$

$$\frac{A(z)}{B(z)} = \frac{1-\phi_1 z}{1-\theta_1 z} = 1 + \sum_{j=1}^{\infty} (\theta_1 - \phi_1) \theta_1^{j-1} z^j$$

$$\Rightarrow Y_t = \frac{B(L)}{A(L)} \varepsilon_t = \varepsilon_t + \sum_{j=1}^{\infty} (\phi_1 - \theta_1) \phi_1^{j-1} \varepsilon_{t-j}, \varepsilon_t = \frac{A(L)}{B(L)} Y_t = Y_t + \sum_{j=1}^{\infty} (\theta_1 - \phi_1) \theta_1^{j-1} Y_{t-j}$$

$$\Rightarrow X_t = Y_t + \frac{\phi_0}{1-\phi_1} = \frac{\phi_0}{1-\phi_1} + \varepsilon_t + \sum_{j=1}^{\infty} (\phi_1 - \theta_1) \phi_1^{j-1} \varepsilon_{t-j}$$

$$\varepsilon_t = X_t - \frac{\phi_0}{1-\phi_1} + \sum_{j=1}^{\infty} (\theta_1 - \phi_1) \theta_1^{j-1} \left(X_{t-j} - \frac{\phi_0}{1-\phi_1} \right)$$

(2)

$$EX_t = EY_t + \frac{\phi_0}{1-\phi_1} = \frac{\phi_0}{1-\phi_1}$$

对于 Y_t, 其自协方差函数为

$$\gamma_k = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k}, \psi_0 = 1, \psi_j = (\phi_1 - \theta_1) \phi_1^{j-1}, j \geq 1$$

$$\gamma_0 = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma^2 \left(1 + (\phi_1 - \theta_1)^2 \sum_{j=1}^{\infty} \phi_1^{2j-2} \right) = \sigma^2 \left(1 + \frac{(\phi_1 - \theta_1)^2}{1-\phi_1^2} \right)$$

$$\gamma_k = \sigma^2 \left((\phi_1 - \theta_1) \phi_1^{k-1} + (\phi_1 - \theta_1)^2 \sum_{j=1}^{\infty} \phi_1^{k+2j-2} \right) = \sigma^2 \left((\phi_1 - \theta_1) \phi_1^{k-1} + \frac{(\phi_1 - \theta_1)^2 \phi_1^k}{1-\phi_1^2} \right), k \geq 1$$

X_t 的自协方差函数 $\gamma'_k = \gamma_k$.

4 (1)

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, EX_t \varepsilon_{t-k} = \begin{cases} \sigma^2 \psi_k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$\psi_j = a_1 \psi_{j-1} + b_j \Rightarrow \psi_0 = 1, \psi_1 = 1.5, \psi_j = \frac{9}{5} \left(\frac{4}{5} \right)^{j-2}, j \geq 2$$

$$\gamma_0 = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 = 12.25 \sigma^2$$

$$X_t = 0.8 X_{t-1} + \varepsilon_t + 0.7 \varepsilon_{t-1} + 0.6 \varepsilon_{t-2}$$

两边同乘 X_{t-1}, 取期望

$$\gamma_1 = 0.8 \gamma_0 + (0.7 \psi_0 + 0.6 \psi_1) \sigma^2 \Rightarrow \rho_1 = 0.8 + (0.7 \psi_0 + 0.6 \psi_1) / 12.25$$

两边同乘 X_{t-2}, 取期望

$$\gamma_2 = 0.8\gamma_1 + 0.6\psi_1\sigma^2 \Rightarrow \rho_2 = 0.8\rho_1 + 0.6\psi_1/12.25$$

两边同乘 $X_{t-k}, k \geq 3$, 取期望

$$\gamma_k = 0.8\gamma_{k-1} \Rightarrow \rho_k = 0.8\rho_{k-1}, k \geq 3$$

(2)

$$A(z) = 1 - 0.8z, B(z) = 1 + 0.7z + 0.6z^2$$

$$A(z) = 0 \Rightarrow z = 1.25, B(z) = 0 \Rightarrow |z_1| = |z_2| = \sqrt{\frac{5}{3}}$$

序列是因果可逆的。

(3)

$$\varepsilon_t = \frac{A(L)}{B(L)} X_t$$

设 $\frac{A(z)}{B(z)} = \sum_{j=0}^{\infty} \xi_j z^j$

$$\left(1 + b_1 z + b_2 z^2\right) \sum_{j=0}^{\infty} \xi_j z^j = 1 - a_1 z$$

$$\Rightarrow \xi_0 + (\xi_1 + b_1\xi_0) z + \sum_{j=2}^{\infty} (\xi_j + b_1\xi_{j-1} + b_2\xi_{j-2}) z^j = 1 - a_1 z$$

$$\Rightarrow \xi_0 = 1, \xi_1 = -a_1 - b_1\xi_0 = -1.5, \xi_2 = 0.45, \xi_j = -b_1\xi_{j-1} - b_2\xi_{j-2}, j \geq 2$$

记 α, β 为 $B\left(\frac{1}{z}\right) = 0$ 的两个根, 令 $c_1 = \frac{\beta\xi_1 - \xi_2}{\beta - \alpha} = \frac{0.45 + 1.5\beta}{\alpha - \beta}, c_2 = \frac{\xi_2 - \alpha\xi_1}{\beta - \alpha} = \frac{0.45 + 1.5\alpha}{\beta - \alpha}$, 则

$$\xi_j = c_1\alpha^{j-1} + c_2\beta^{j-1}, j \geq 3$$

$$\varepsilon_t = \sum_{j=0}^{\infty} \xi_j X_{t-j} = X_t - 1.5X_{t-1} + 0.45X_{t-2} + \sum_{j=3}^{\infty} \left(c_1\alpha^{j-1} + c_2\beta^{j-1}\right) X_{t-j}$$