1. (a) For any $\epsilon > 0$ and $n \in \mathbb{Z}$, we have

$$P(|Z_n| > \epsilon^{-1/2}) \leq \text{Var}(Z_n)\epsilon = \epsilon,$$

by Chebyshev inequality. By definition, $Z_n = O_p(1)$.
Notice that $X_n = m + \sigma_n Z_n = m + \sigma_n O_p(1)$ and $\sigma_n^2 \to 0$, then $X_n = m + o_p(1)$.
(b) By Taylor expansion, we have

$$f(X_n) = f(m) + f^{'}(m)(X_n - m) + o_p(|X_n - m|).$$

Then, $Y_n - Z_n = (f^{'}(m)\sigma_n)^{-1}(o_p(|X_n - m|)) = o_p(1)$ by the fact $X_n = m + o_p(1)$ and $\sigma_n^2 \to 0$.

(c) Suppose there exists a random varaible Z such that $Z_n \xrightarrow{P} Z$. By the definition, $o_p(1) \xrightarrow{P} 0$. Then $Y_n = Z_n + o_p(1) \xrightarrow{p} Z$. Suppose there exists a random varaible Z such that $Z_n \xrightarrow{d} Z$. $Y_n = Z_n + o_p(1) \xrightarrow{d} Z$ by the Slutsky Theorem.
(d) By the Central Limit Theorem , we have

$$\sqrt{n}(\frac{S_n}{n} - p) \xrightarrow{d} N(0, p(1-p)).$$

By (c),

$$f(\frac{S_n}{n}) \xrightarrow{d} N\left(f(p), \frac{p(1-p)(f^{'}(p))^2}{n}\right).$$

2. By the Slutsky Theorem, $X_n \xrightarrow{d} \mu$. Since μ is a constant, $X_n \xrightarrow{P} \mu$.

3. (a) Sufficiency: By the condition, we have

$$\frac{X_n - \mu_n}{\sigma_n} = \frac{X_n - \tilde{\mu}_n}{\tilde{\sigma}_n}\frac{\tilde{\sigma}_n}{\sigma_n} + \frac{\tilde{\mu}_n - \mu_n}{\sigma_n}.$$

It is easy to proof that $N(0,1) \stackrel{d}{=} N(\frac{\tilde{\mu}_n - \mu_n}{\sigma_n}, \left(\frac{\tilde{\sigma}_n}{\sigma_n}\right)^2)$. Thus, $\frac{\tilde{\sigma}_n}{\sigma_n} \to 1$ and $\frac{\tilde{\mu}_n - \mu_n}{\sigma_n} \to 0$.
Necessity: The result follows the Slutsky Theorem and the equation

$$\frac{X_n - \mu_n}{\sigma_n} = \frac{X_n - \tilde{\mu}_n}{\tilde{\sigma}_n}\frac{\tilde{\sigma}_n}{\sigma_n} + \frac{\tilde{\mu}_n - \mu_n}{\sigma_n}.$$

(b) The proof is similar to (a).
(c) The former result follows (b). It is easy to proof that $n^{-1/2}X_n$ is $\text{An}(n^{1/2}, 2)$.

4. Wrong (Deduct 1 point): If $n \to +\infty$, (a) $\bar{X}_n^2 \xrightarrow{d} \mu^2$; (b) $\bar{X}_n^2 \xrightarrow{d} 0$.

Denote $\bar{X}_n = n^{-1}\sum_{j=1}^n X_j$, we have $\bar{X}_n \sim N(\mu, \sigma^2/n) \xrightarrow{d} \mu$. By the Continuous Mapping Theorem, (a) $\bar{X}_n^2 \sim \frac{\sigma^2}{n}\chi_1^2(\lambda)$, where $\lambda = \mu^2$ is the location parameter; (b) $\bar{X}_n^2 \sim \frac{\sigma^2}{n}\chi_1^2$.

Or using delta method to obtain the convergent form of the normal distribution. (a) $\bar{X}_n^2 \sim N(\mu^2, 4\mu\sigma^2/n)$; (b) $\bar{X}_n^2 \sim \frac{\sigma^2}{n}\chi_1^2$.

Moment method or Characteristic function method also works.