1. (1) Let $\{A_i\}_{i=1}^5$ be zero-mean uncorrelated r.v.s, $Var A_i = 1$.
Define

$$X_n = A_1 + A_2 \cos(n\pi/2) + A_3 \cos(n\pi/4) + A_4 \sin(n\pi/2) + A_5 \sin(n\pi/4)$$

then $EX_n = 0$ and $Cov(X_n, X_{n+h}) = f(h)$.
(2) Since $f(4) = 3 > f(0) = 1$, $f(h)$ can not be the autocovariance function of a stationary series.
(3) $\forall n \in N$, consider

$$\Sigma_n = \begin{pmatrix} 1 & 0.4 & & & \\ 0.4 & 1 & 0.4 & & \\ & \ddots & \ddots & \ddots & \\ & & 0.4 & 1 & 0.4 \\ & & & 0.4 & 1 \end{pmatrix}$$

The eigenvalues of Σ_n are

$$\lambda_k = 1 - 2 \times 0.4 \cos(\frac{k\pi}{n+1}) > 0, \quad k = 1, \ldots, n.$$

Σ_n is positive definite.
(4) $\forall n \in N$, consider

$$\Sigma_n = \begin{pmatrix} 1 & 0.6 & & & \\ 0.6 & 1 & 0.6 & & \\ & \ddots & \ddots & \ddots & \\ & & 0.6 & 1 & 0.6 \\ & & & 0.6 & 1 \end{pmatrix}$$

The eigenvalues of Σ_n are

$$\lambda_k = 1 - 2 \times 0.6 \cos(\frac{k\pi}{n+1}), \quad k = 1, \ldots, n.$$

Σ_n is not positive definite.

2. (1)

$$\begin{aligned} Ee^{tZ^2} &= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{\sqrt{1-2t}} \end{aligned}$$

(2) By the independence,

$$Ee^{t(Z_1^2+\cdots+Z_n^2)} = E\prod_{i=1}^n e^{tZ_i^2} = \prod_{i=1}^n Ee^{tZ_i^2} = \frac{1}{(1-2t)^{n/2}}$$

(3) Since Σ is non-singular, it must be positive-definite.

So we have

$$\Sigma = (\Sigma^{1/2})^2,$$

which yields

$$\Sigma^{-1} = (\Sigma^{-1/2})^2,$$

where $\Sigma^{-1/2} = (\Sigma^{1/2})^{-1}$.

Hence,

$$\Sigma^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu}) \sim N(0, I).$$

$$\begin{aligned}(\boldsymbol{X} - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) &= [\Sigma^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu})]'[\Sigma^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu})] \\ &\sim \chi^2(n).\end{aligned}$$

3. (The last two were calculated using MATLAB)

$$\begin{aligned}\phi(t_1, t_2, t_3) &= e^{it^T\mu - \frac{t^T\Sigma t}{2}} \\ &= \exp\{-\frac{(t_1^2 + t_2^2 + t_3^2)}{2}\sigma^2 - t_1t_2\sigma_{12} - t_1t_3\sigma_{13} - t_2t_3\sigma_{23}\}. \\ E(X_1X_2X_3) &= i^{-3}\frac{\partial^3\phi(t_1, t_2, t_3)}{\partial t_1\partial t_2\partial t_3}|_{t_1=0,t_2=0,t_3=0} \\ &= 0. \\ E(X_1^2X_2^2X_3^2) &= i^{-6}\frac{\partial^6\phi(t_1, t_2, t_3)}{\partial t_1^2\partial t_2^2\partial t_3^2}|_{t_1=0,t_2=0,t_3=0} \\ &= \sigma^6 + 2\sigma^2(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) + 8\sigma_{12}\sigma_{13}\sigma_{23}. \\ E[(X_1^2 - \sigma^2)(X_2^2 - \sigma^2)(X_3^2 - \sigma^2)] &= E[X_1^2X_2^2X_3^2 + \sigma^4(X_1^2 + X_2^2 + X_3^2) \\ &\quad - \sigma^2(X_1^2X_2^2 + X_1^2X_3^2 + X_2^2X_3^2) - \sigma^6] \\ &= 8\sigma_{12}\sigma_{13}\sigma_{23}.\end{aligned}$$

4.

$$\begin{aligned}F(y_1, y_2) &= P(X_1 \le y_1, |X_2| \le y_2, X_1 \ge 0) + P(X_1 \le y_1, -|X_2| \le y_2, X_1 < 0) \\ &= \begin{cases}(\Phi(y_1) - 1/2)(2\Phi(y_2) - 1) + 1/2, & y_1 > 0, y_2 > 0 \\ \Phi(y_2), & y_1 > 0, y_2 < 0 \\ \Phi(y_1), & y_1 \le 0, y_2 \ge 0 \\ 2\Phi(y_1)\Phi(y_2), & y_1 \le 0, y_2 < 0 \\ 0, & else\end{cases}\end{aligned}$$

Hence, $F_1(y_1) = F(y_1, \infty) = \Phi(y_1)$, $F_2(y_2) = F(\infty, y_2) = \Phi(y_2)$.

Since $f(y_1, y_2) = 0$ when $y_1y_2 < 0$, (Y_1, Y_2) must be non-normal as the support of a bivariate normal distribution (nondegenerate) is $\mathbf{R}^2$.

5. a.

$$EX_t = \begin{cases}E\varepsilon_t, & t \text{ is even} \\ E(\varepsilon_{t-1}^2 - 1)/\sqrt{2}, & t \text{ is odd}\end{cases} = 0.$$

$$Cov(X_{t_1}, X_{t_2}) = \begin{cases}1, & t_1 = t_2, \\ 0, & else.\end{cases}$$

b.
n is even: $E(X_{n+1}|X_1,\dots,X_n) = E((\varepsilon_n^2-1)/\sqrt{2}|X_n) = (X_n^2-1)/\sqrt{2}$.
n is odd: $E(X_{n+1}|X_1,\dots,X_n) = E(\varepsilon_{n+1}|X_n,\dots,X_1) = 0$.

6. If $s = t$

$$\begin{aligned} P(X_t \le x, X_s \le y) &= P(X_t \le \min\{x,y\}) \\ &= P(X_{t-12} + \varepsilon_t \le \min\{x,y\}) \\ &= \Phi(\frac{\min\{x,y\}}{\sqrt{\lceil t/12 \rceil}\sigma}). \end{aligned}$$

Assume $s > t$, then

$$\begin{aligned} P(X_t \le x, X_s \le y) &= P(X_{t-12} + \varepsilon_t \le x, X_{s-12} + \varepsilon_s \le y) \\ &= \cdots \\ &= P\Big(\sum_{k=0}^{\lceil t/12 \rceil} \varepsilon_{(t \ mod \ 12)+12k} \le x, \sum_{k=0}^{\lceil s/12 \rceil} \varepsilon_{(s \ mod \ 12)+12k} \le y\Big) \\ &= \begin{cases} G(x,y), & (t \ mod \ 12) = (s \ mod \ 12) \\ \Phi(\frac{x}{\sqrt{\lceil t/12 \rceil}\sigma})\Phi(\frac{y}{\sqrt{\lceil s/12 \rceil}\sigma}), & (t \ mod \ 12) \neq (s \ mod \ 12) \end{cases} \end{aligned}$$

where $G(\cdot,\cdot)$ is the cdf of $N(0,\Sigma)$ and

$$\Sigma = \sigma^2 \begin{pmatrix} \lceil t/12 \rceil & \lceil t/12 \rceil \\ \lceil t/12 \rceil & \lceil s/12 \rceil \end{pmatrix}.$$

7.

$$\Sigma^{1/2} = \begin{pmatrix} 2\sqrt{6}/3 & \sqrt{6}/3 & \sqrt{6}/3 \\ \sqrt{6}/3 & \sqrt{6}/6 & \sqrt{6}/6 \\ \sqrt{6}/3 & \sqrt{6}/6 & \sqrt{6}/6 \end{pmatrix}$$

solvng $\Sigma^{1/2}a = 0$, we have $Var(\sum_{i=1}^3 a_i X_i) = a^T \Sigma a = 0$.

8. (a)

$$\begin{aligned} E\exp(\sum_{i=1}^n \lambda_i X_i) &= E\exp(\sum_{i=1}^n \lambda_i (Z_i + \theta Z_{i-1})) \\ &= E\exp(\sum_{i=1}^{n-1} (\lambda_i + \theta\lambda_{i-1}) Z_i + \lambda_n Z_n + \theta\lambda_1 Z_0) \\ &= m(\lambda_n) m(\theta\lambda_1) \prod_{i=1}^{n-1} m(\lambda_i + \theta\lambda_{i-1}). \end{aligned}$$

(b)
By the result in (a), $\forall n \in N,\ t_1,\dots,t_n$, the moment-generating function of $(X_{t_1},\dots,X_{t_n})$ is irrelevant to $(t_1,\dots,t_n)$.

Hence, $\forall k \in Z$, $(X_{t_1},\dots,X_{t_n}) \overset{d}{=} (X_{t_1+k},\dots,X_{t_n+k})$.