1. Since $\sum\limits_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{i(h-k)\lambda} d\lambda \\
&= \gamma(h) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k \neq h} \gamma(k) e^{i(h-k)\lambda} d\lambda \\
&= \gamma(h) + \frac{1}{2\pi} \sum_{k \neq h} \gamma(k) \int_{-\pi}^{\pi} \cos\big((h-k)\lambda\big) + i \sin\big((h-k)\lambda\big) d\lambda \\
&= \gamma(h)
\end{aligned}$$

2. Observe that when $h \neq 0$

$$\begin{aligned}
\gamma(h) &= \frac{\sin(ah)}{h} \\
&= \frac{e^{iah} - e^{-iah}}{2ih} \\
&= \int_{-a}^{a} \frac{e^{ivh}}{2} dv \\
&= \int_{-\pi}^{\pi} \frac{I_{[-a,a]}(v)}{2} e^{ivh} dv
\end{aligned}$$

we have

$$f(v) = \frac{I_{[-a,a]}(v)}{2}$$

hence, the spectral distribution function is

$$F(v) = \int_{-\pi}^{\lambda} f(v) dv = \frac{I_{[-a,a]}(\lambda)}{2} (\lambda + a) + I_{(a,\pi]}(\lambda) a$$

3. Denote by $Z_t = A\cos(\pi t/3) + B\sin(\pi t/3)$
$\forall s, t \in \mathbb{Z}$, the autocovariance function of X_t is

$$\begin{aligned}
\gamma(t,s) &= Cov(X_t, X_s) \\
&= Cov(Z_t + Y_t, Z_s + Y_s) \\
&= Cov(Z_t, Z_s) + Cov(Y_t, Y_s) \\
&= v^2(\cos(\pi t/3)\cos(\pi s/3) + \sin(\pi t/3)\sin(\pi s/3)) + Cov(Y_t, Y_s) \\
&= \begin{cases} v^2 + 7.25\sigma^2, & s = t \\ v^2/2 + 2.5\sigma^2, & |s-t| = 1 \\ v^2\cos((s-t)\pi/3), & else \end{cases}
\end{aligned}$$

That is

$$\gamma(h) = v^2 \cos\big(\pi h/3\big) + \sigma^2(I_{\{0\}}(h) + 6.25 I_{\{0,1\}}(h))$$

The spectral density of Y_t is

$$f_Y(\lambda) = \frac{\sigma^2}{2\pi}(7.25 + 5\cos\lambda)$$

Hence, $F_Y(\lambda) = \frac{\sigma^2}{2\pi}[7.25(\lambda+\pi) + 5\sin(\lambda)]$

The spectral distribution function of Z_t is

$$F_Z(\lambda) = \frac{v^2}{2}[I_{[-\frac{\pi}{3},\pi]}(\lambda) + I_{[\frac{\pi}{3},\pi]}(\lambda)]$$

So the spectral distribution function of X_t is

$$\begin{aligned}
F(\lambda) &= F_Z(\lambda) + F_Y(\lambda) \\
&= \frac{v^2}{2}[I_{[-\frac{\pi}{3},\pi]}(\lambda) + I_{[\frac{\pi}{3},\pi]}(\lambda)] + \frac{\sigma^2}{2\pi}[7.25(\lambda+\pi) + 5\sin(\lambda)]
\end{aligned}$$

4. Since $|a| < 1$, $\{X_t\}$ is stationary, hence

$$EX_t = 0, \quad \gamma(0) = a^2\gamma(0) + \sigma^2$$

which yields that $\gamma(0) = \frac{\sigma^2}{1-a^2}$

$$\begin{aligned}
\gamma(h) &= E(X_{t+h}X_t) \\
&= E[(\varepsilon_{t+h} + a\varepsilon_{t+h-1} + \cdots + a^h\varepsilon_t + \cdots) \\
&(\varepsilon_t + a\varepsilon_{t-1} + \cdots)] \\
&= a^h(1 + a^2 + a^4 + \cdots)\sigma^2 \\
&= \frac{a^h}{1-a^2}\sigma^2
\end{aligned}$$

So the autocorrelation is $\rho(h) = a^h$

The spectral density is

$$\begin{aligned}
f(\lambda) &= \sum_{h=-\infty}^{\infty} \gamma(h)e^{-ih\lambda} \\
&= \frac{\sigma^2}{2\pi}(1 - 2a\cos\lambda + a^2)^{-1}
\end{aligned}$$