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1. For a many–electron system, the Hamiltonian is given by

$$\hat{H} = \sum_{i=1}^{N} \left[\frac{\hat{\boldsymbol{p}}_i^2}{2m} + V(\hat{\boldsymbol{r}}_i) \right] + \frac{1}{2} \sum_{i \neq j} \frac{e^2}{|\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j|}. \tag{1}$$

Use a set of single particle basis $\{|\varphi_i, \sigma\rangle\}$ to express the Hamiltonian in the second quantization form. Here $\langle \boldsymbol{r}|\varphi_i\rangle \equiv \varphi_i(\boldsymbol{r})$ is the spatial orbital and $\sigma = \pm$ presents the spin degree of freedom. Note that the spin degree of freedom has to be integrated.

2. Consider a system with N non-interacting neutrons moving in an external potential $V_0\delta(x)$. The Hamiltonian reads

$$\hat{H} = \sum_{i=1}^{N} \left[\frac{\hat{p}_i^2}{2m} + V_0 \delta(\hat{x}_i) \right]. \tag{2}$$

Define the single electron state via

$$\hat{a}_{k\sigma}^\dagger |0\rangle = |k\sigma\rangle. \tag{3}$$

It is the mutual eigenstates of momentum and spin, i.e.,

$$\hat{p}|k\sigma\rangle = \hbar k|k\sigma\rangle, \quad \hat{S}|k\sigma\rangle = \frac{\hbar\sigma}{2}|k\sigma\rangle, \tag{4}$$

and in the position representation, we have

$$\langle x|k\sigma\rangle = \frac{1}{\sqrt{L}} e^{ikx} \tag{5}$$

with $k = \frac{2\pi n}{L}$ $(n = 0, \pm 1, \pm 2, \cdots)$ and

$$\langle k\sigma|k'\sigma\rangle = \frac{1}{L} \int_{-L/2}^{L/2} e^{-i(k-k')x} \mathrm{d}x = \delta_{kk'}. \tag{6}$$

(a) Express the Hamiltonian in terms of the creation and annihilation operators.

(b) Treat V_0 as a small quantity, and use the perturbation theory to calculate the ground state energy of the system up to the lowest non-vanishing order.