September 25, 2023

1. *Exercise E2.3.1.* Equation (2.3.6d) is corrected to

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}.$$

2. (Optional) Consider an electron moving in the presence of a time–independent electromagnetic field. The Hamiltonian is given by

$$\hat{H} = \frac{1}{2m}\left[\hat{\vec{p}} - \frac{e}{c}\vec{A}(\hat{\vec{x}})\right]^2 + e\varphi(\hat{\vec{x}}), \tag{1}$$

where $e \equiv -|e|$ is the charge of an electron, c is the speed of light, and $\vec{A}(\vec{x})$ and $\varphi(\vec{x})$ are the vector and scalar potentials of the external electromagnetic field, respectively. Define the kinetic momentum

$$\hat{\vec{\pi}} \equiv \hat{\vec{p}} - \frac{e}{c}\vec{A}(\hat{\vec{x}}), \tag{2}$$

and then the Hamiltonian is recast to

$$\hat{H} = \frac{\hat{\vec{\pi}}^2}{2m} + e\varphi(\hat{\vec{x}}). \tag{3}$$

Show that in the Heisenberg picture, we have

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{\vec{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\vec{\pi}} = e\left[\vec{E} + \frac{1}{2c}\left(\frac{\mathrm{d}\hat{\vec{x}}}{\mathrm{d}t} \times \vec{B} - \vec{B} \times \frac{\mathrm{d}\hat{\vec{x}}}{\mathrm{d}t}\right)\right], \tag{4}$$

where

$$\vec{E} = -\nabla\varphi \tag{5}$$

and

$$\vec{B} = \nabla \times \vec{A} \tag{6}$$

are the electric and magnetic field, respectively. Equation (4) indicates the quantum version of the Lorentz force.