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1. Suppose $\{|\varphi_n^k\rangle\}_{n=1}^{l_k}$ is the basis of k-th irreducible representation of a group $\mathbb{G} \equiv \{\hat{\Gamma}_i\}_{i=1}^{h}$. Define the projection operator being

$$\hat{P}_{mn}^k \equiv \frac{l_k}{h} \sum_{i=1}^{h} \Lambda_{mn}^k(\hat{\Gamma}_i)^* \hat{\Gamma}_i. \tag{1}$$

Show that for an arbitrary state $|\psi\rangle$, $\hat{P}_{mn}^k|\psi\rangle$ is one of the basis of k-th irreducible representation, i.e., to prove

$$\hat{\Gamma}_i\big(\hat{P}_{mn}^k|\psi\rangle\big) = \sum_{m'} \big(\hat{P}_{m'n}^k|\psi\rangle\big)\Lambda_{m'm}^k. \tag{2}$$

And show that

$$\hat{P}_{mm'}^k|\varphi_n^k\rangle = \delta_{m'n}|\varphi_m^k\rangle. \tag{3}$$

Here, one may use the identity,

$$\hat{\Gamma}_i|\varphi_m^k\rangle = \sum_n |\varphi_n^k\rangle\langle\varphi_n^k|\hat{\Gamma}_i|\varphi_m^i\rangle = \sum_n |\varphi_n^k\rangle\Lambda_{nm}^k(\hat{\Gamma}_i). \tag{4}$$

2. Construct all possible character tables for a group with four elements.