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1. Argue that for a many electron system, the total wave function can de decomposed as product of spacial and spin parts, i.e., $\Psi(\boldsymbol{r}_1 s_1, \boldsymbol{r}_2 s_2, \cdots, \boldsymbol{r}_N s_N) = \Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)\sigma(s_1, s_2, \cdots, s_N)$. And further discuss that the spacial part $\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)$ is symmetric or antisymmetric under the exchange of two electrons. (*Hint: Consider the two–electron case and generalize the results. No rigorous proof is needed.*)

2. Jordan–Wigner transformation gives a one–to–one matrix representation of fermion creation and annihilation operators. Define

$$\hat{d}_i \equiv \left(\prod_{j=1}^{i-1} \otimes \hat{\sigma}_j^z\right) \otimes \hat{\sigma}^- \otimes \left(\prod_{j=i+1}^{N} \otimes \mathbf{I}\right), \tag{1}$$

where N is the total number of single electron states, $\hat{\sigma}^{\pm} \equiv \frac{1}{2}(\hat{\sigma}^x \pm i\hat{\sigma}^y)$, and

$$\hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2}$$

Here the direct product is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} & \cdots & A_{1n}\mathbf{B} \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} & \cdots & A_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}\mathbf{B} & A_{n2}\mathbf{B} & \cdots & A_{nn}\mathbf{B} \end{pmatrix}. \tag{3}$$

For $N = 2$, write down $\hat{d}_{i=1,2}$ and verify that

$$\{\hat{d}_i, \hat{d}_j^\dagger\} = \delta_{ij}, \ \{\hat{d}_i, \hat{d}_j\} = 0. \tag{4}$$

Further define $\hat{n}_i \equiv \hat{a}_i^\dagger \hat{a}_i$ and write down the matrix form of the single impurity Anderson model,

$$\hat{H} = \epsilon_1 \hat{n}_1 + \epsilon_2 \hat{n}_2 + U \hat{n}_1 \hat{n}_2. \tag{5}$$