November 8, 2023

1. An electron moves in the presence of a uniform magnetic field with vector potential being

$$\hat{\boldsymbol{A}} = -B_0\hat{y}\mathbf{i}.$$

From the principle of electromagnetism, the Hamiltonian reads

$$\hat{H} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{e}{c}\hat{\boldsymbol{A}}\right)^2 = \frac{1}{2m}\left[\left(\hat{p}_x + \frac{eB_0}{c}\hat{y}\right)^2 + \hat{p}_y^2 + \hat{p}_z^2\right].$$

By assuming the eigenfunction of the form,

$$\psi(x, y, z) = e^{i(k_x x + k_z z)} f(y),$$

prove that the eigenvalues are

$$E(k_z, j) = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega_0\left(j + \frac{1}{2}\right),$$

with $\omega_0 \equiv |eB_0|/mc$ and $j \in \mathbb{Z}$.

2. After Schrödinger proposes the wave mechanics, people immediately start to seize the relativistic version of it. The first attempt is the Klein–Gordon equation. We start from the energy–momentum relation for a relativistic particle with mass m,

$$E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4. \tag{1}$$

For simplicity, we set $\hbar = c = 1$, then the above equation can be written as

$$E^2 = \boldsymbol{p}^2 + m^2. \tag{2}$$

By quantizing with

$$E \to i\frac{\partial}{\partial t}, \quad \boldsymbol{p} \to -i\nabla, \tag{3}$$

show that the relativistic wave equation reads

$$(\partial_t^2 - \nabla^2 + m^2)\psi(\boldsymbol{r}, t) = 0. \tag{4}$$

Further prove that

$$\rho \equiv \frac{i}{2m}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \tag{5}$$

and

$$\boldsymbol{J} \equiv \frac{1}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*) \tag{6}$$

satisfy the continuity equation. Based on the result, argue that why the Klein–Gordon equation cannot be a function of a probability wave, that is, why it cannot be a correct relativistic version of Schrödinger equation.