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1. The Gaussian wave packet presents great number of unique properties in quantum physics. The wave function of a Gaussian wave packet is given by

$$\Phi(x) = \langle x|\Phi\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left(ikx - \frac{x^2}{2d^2}\right), \tag{1}$$

where d and k are two real parameters.

   (a) Show that $\Phi(x)$ is normalized.

   (b) Show that the wave function in the momentum representation reads

$$\langle p|\Phi\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left[-\frac{(p-\hbar k)^2 d^2}{2\hbar^2}\right]. \tag{2}$$

   (c) Try to explain the physical meaning of d and k.

   (d) For a one dimensional free particle with Hamiltonian being

$$\hat{H} = \frac{\hat{p}^2}{2m}, \tag{3}$$

   if the initial state is a Gaussian wave packet, i.e., $|\Psi(t=0)\rangle = |\Phi\rangle$, solve the Schrödinger equation to obtain $|\Psi(t>0)\rangle$.

The Gaussian integral might be useful, which is,

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}, \tag{4}$$

with a, b complex.