September 4, 2023

1. Use the Fourier series as an example. The Fourier series of a function $f(t)$ with period T is defined as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}, \tag{1}$$

with $\omega_n \equiv 2\pi n/T$ and the inner product is

$$\langle f|g\rangle \equiv \frac{1}{T}\int_0^T \mathrm{d}t\, f^*(t)g(t). \tag{2}$$

Prove that the basis $|n\rangle \equiv e^{i\omega_n t}$ is orthonormal,

$$\langle n|m\rangle = \delta_{nm}, \tag{3}$$

and the combination coefficients can be obtained by

$$c_n = \langle n|f\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\, f(t)e^{-i\omega_n t}. \tag{4}$$

2. Prove that the trace and determinant of a matrix $\mathbf{A}$ are invariant under a unitary transformation, that is, prove

$$\mathrm{Tr}(\mathbf{U}\mathbf{A}\mathbf{U}^\dagger) = \mathrm{Tr}(\mathbf{A}) \quad \text{and} \quad \det(\mathbf{U}\mathbf{A}\mathbf{U}^\dagger) = \det(\mathbf{A}). \tag{5}$$

3. Find the eigenvalues and normalized eigenvectors of the matrix

$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{6}$$