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1. For hydrogen–like systems, the energy levels are given by

$$E_n = -\frac{Z^2}{n^2}\mathcal{R}, \tag{1}$$

with $\mathcal{R}$ being the Rydberg constant, Z the atomic number, and n the principal quantum number. It is noticed that the eigenvalues of hydrogen–like systems are greatly degenerate, indicating that the Hamiltonian,

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m_e} - \frac{Ze^2}{r}, \tag{2}$$

has more internal symmetries or conserved dynamic quantities. The symmetry arises from nothing but the Runge–Lenz (RL) vector you have encountered in the classical mechanics. The RL vector is defined as

$$\hat{\boldsymbol{M}} \equiv \frac{1}{2m_e}(\hat{\boldsymbol{p}} \times \hat{\boldsymbol{L}} - \hat{\boldsymbol{L}} \times \hat{\boldsymbol{p}}) - \frac{Ze^2}{r}\hat{\boldsymbol{r}}, \tag{3}$$

where $\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$ is the angular momentum operator. Show that

$$[\hat{\boldsymbol{M}}, \hat{H}] = 0 \tag{4}$$

and

$$\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{M}} = \hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{L}} = 0. \tag{5}$$

2. Exercise *E5.4.1-5.4.2* in textbook.