November 15, 2023

1. Using the algebra of angular momentum, one may prove the projection theorem, reading

$$\langle \alpha', j, m_j' | \hat{\boldsymbol{V}} | \alpha, j, m_j \rangle = \langle \alpha', j, m_j' | \frac{\hat{\boldsymbol{J}}}{\hat{J}^2} \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{V}} | \alpha, j, m_j \rangle. \tag{1}$$

   Here $\hat{\boldsymbol{V}}$ is an arbitrary vector operator, $\hat{\boldsymbol{J}}$ is diagonalized within $\{|j, m_j\rangle\}$, and $\{\alpha\}$ represents other quantum numbers. The theorem states that the average of $\hat{\boldsymbol{V}}$ can be effectively replaced by its projection along $\hat{\boldsymbol{J}}$. Consider the spin–orbit coupling, the total angular momentum is defined via the addition of the orbital angular momentum and the spin angular momentum,

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}. \tag{2}$$

   The irreducible states are chosen as

$$|l, s; j, m_j\rangle \tag{3}$$

   being a simultaneous eigenstate of $\hat{\boldsymbol{J}}^2$, $\hat{J}_z$, $\hat{\boldsymbol{L}}^2$, and $\hat{\boldsymbol{S}}^2$. Using the projection theorem, evaluate

$$\langle l, s; j, m_j | \hat{W} | l, s; j, m_j \rangle, \tag{4}$$

   where

$$\hat{W} \equiv \frac{e}{2m_e} (\hat{\boldsymbol{L}} + g_{\mathrm{S}} \hat{\boldsymbol{S}}) \cdot \boldsymbol{B} \tag{5}$$

   with g_{S} being the Landé factor and $\boldsymbol{B}$ the external constant magnetic field.

2. (*For your information*) This part is a simple review on the relativistic mechanics. In the special relativity, the movement of a particle is described by the time–space coordinate

$$(t, \boldsymbol{x}) \equiv x^{\mu=0,1,2,3}. \tag{6}$$

Here we set $c \equiv 1$. x^μ is a vector in the Minkowski space with the inner product defined as

$$x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu \equiv x_\mu x^\mu = -t^2 + \boldsymbol{x}^2. \tag{7}$$

Here we use the Einstein summation convention, i.e., the repeated indices are summed over. The metric tensor $\eta_{\mu\nu}$ is defined as

$$\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\} \tag{8}$$

and

$$x_\mu \equiv \eta_{\mu\nu} x^\nu. \tag{9}$$

Define the proper time of a particle as

$$\mathrm{d}\tau^2 \equiv -\mathrm{d}x^2 = -\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2. \tag{10}$$

For a relativistic particle, we have $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = 1$, and its proper time is zero. The transformation in the Minkowski space is the Lorentz transformation, defined via

$$x'^\mu \equiv \Lambda^\mu{}_\nu x^\nu, \tag{11}$$

which conserves the inner product x^2 unchanged. This is also equivalent to

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}. \tag{12}$$

It is easy to check $\mathrm{d}\tau^2$ is also invariant under a Lorentz transformation. The momentum of a particle is defined as

$$p^\mu \equiv m \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \tag{13}$$

where m is the rest mass. Its components read

$$p^0 = m \frac{\mathrm{d}t}{\mathrm{d}\tau} \equiv m\gamma \equiv E \tag{14}$$

and

$$\boldsymbol{p} = m\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = m\gamma\boldsymbol{v}. \tag{15}$$

Here we introduce the Lorentz contraction factor,

$$\gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t})^2}} = \frac{1}{\sqrt{1-\boldsymbol{v}^2}}, \tag{16}$$

and the velocity vector $\boldsymbol{v} \equiv \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \dot{\boldsymbol{x}}$. Thus we have

$$p^2 = m^2\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -m^2. \tag{17}$$

On the other hand

$$p^2 = -E^2 + \boldsymbol{p}^2, \tag{18}$$

then we have the energy–momentum relation

$$E^2 = m^2 + \boldsymbol{p}^2. \tag{19}$$